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SHAPING FILTER MODELS FOR NONSTATIONARY RANDOM PROCESSES

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ABSTRACT

This research is concerned with the synthesis of a linear system called a shaping filter which transforms a white noise input into a possibly nonstationary output random process with a given autocorrelation function. The determination of a shaping filter provides a solution of the so-called factorization problem.

Conditions are developed which, if satisfied by an autocorrelation function, guarantee a solution of the factorization problem. The method of factorization requires the formulation of a matrix Riccati equation. Coefficients of the shaping filter are easily related to a solution of the Riccati equation. In order to formulate the Riccati equation, new results concerning the mean-square differentiability of a random process are developed and proved. Autocorrelation functions which admit factorization are characterized by easily applied criteria which do not depend explicitly on the non-negative definite condition, a condition necessarily satisfied by any autocorrelation function.

An upper bound is derived which, if satisfied by the initial condition for the Riccati equation, insures that the solution of the Riccati equation, and hence of the factorization problem, is defined globally. If the condition for a global solution of the Riccati equation is satisfied and if the given autocorrelation function is bounded, then the shaping filter will be stable in an appropriately defined sense.

In order to broaden the class of autocorrelation functions which admit factorization, cases are considered for which the Riccati equation has an isolated singular point, and for which the Riccati equation is undefined everywhere. In the former case, sufficient conditions are developed which insure that the Riccati equation has a solution continuous at the singular point. In the latter case, the factorization problem has an algebraic solution.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

1.1 Motivation

It is often convenient to model a random process as the result of a linear filtering operation on stationary white noise. Such a representation has proved invaluable when applied to many signal processing problems, especially those associated with the theory of filtering and estimation of random signals [1.1 - 1.6]. In such applications, the given random signal or process is frequently specified only by its autocorrelation function. For example, optimal estimation problems involving a minimum mean-square error criterion invariably may be stated in terms of appropriate autocorrelation and cross-correlations of given random variables or process [1.7]. In such cases, the statistical description of a random process given solely in terms of its second-order properties, i.e., its autocorrelation function, clearly suffices for the purpose of solving the estimation problem.

A more physical description of a random process with a given autocorrelation function employs a linear system called a shaping filter which transforms stationary white noise into a process having the given autocorrelation function. By introducing shaping filters, great simplicity has been achieved in the formulation and solution of many estimation, filtering, and prediction problems. Wiener [1.1] used the shaping filter concept implicitly, and more recently, Darlington [1.9], Kalman and Bucy [1.6], and others used it explicitly. Clearly, in order to understand the generality or the limitations of a shaping filter description of a random process, one must investigate the possibility of transforming a statistical description of the process into a shaping filter description. The so-called factorization problem, concerned with determining a shaping filter from a given autocorrelation function, is the primary subject of the present investigation.

1.2 Problem Formulation and Background

For stationary random processes, the factorization problem has well-known frequency domain solutions. General solutions are given by Wiener [1.1], and Doob [1.8], and solutions for the case of a rational power spectral density are given by Wiener [1.1], and Bode and Shannon [1.2]. In the latter case, the shaping filters may be realized by the interconnection of a finite number of lumped elements. For purposes of simulation or computation, the latter case has great practical significance.

If the given autocorrelation function corresponds to a nonstationary random process, the factorization problem becomes particularly interesting and challenging. Although previous investigations of the nonstationary factorization problem have met with varying degrees of success, the problem has not been solved in general. A common assumption among these investigations, as well as the present one, is that the random processes under consideration are nonstationary analogues of those considered by Bode and Shannon. In other words, a shaping filter is represented either by a single linear differential equation of order n, or more generally, by a set of first-order linear differential equations in "state-variable" form with time-varying coefficients.

Among the first and most significant contributions to the solution of the factorization problem is that of Darlington [1.9]. Darlington assumed the existence of a single n-th order differential equation model for the shaping filter. Using the algebra of time-varying differential operators, and a method analogous to that employed by Bode and Shannon, he exhibited global solutions of the factorization problem, provided that the time variations involved were suitably defined and restricted. The coefficients of the shaping filter may be obtained from the solutions of a related linear differential equation.

Batkov [1.10] , at about the same time as Darlington, proposed an algebraic solution of the factorization problem. However, according to Stear [1.11] , Batkov's method fails except in certain special cases. R.P. Webb, et. al. [1.12], considered a state-variable solution of the problem. Their solution too appears to be invalid except in special cases.

Other relevant investigations are those of Kalman [1.13], Stear [1.11], and Anderson [1.14], and are concerned with state-variable formulations. Although Kalman did not solve the factorization problem, he was able to establish a formal definition of the problem. The results of Stear and Anderson, although derived by different methods, are similar and appear to provide a first step in demonstrating the existence of a factorization for the general nonstationary case.

A shortcoming common to all these investigations in that they fail, in varying degrees, to relate the existence of a shaping filter to appropriate properties of the autocorrelation function. That is, the question of how to characterize the class of autocorrelation functions which admit factorization has thus far gone unanswered. The present investigation may be regarded as an attempt to develop a formal realizability theory for shaping filters having a state-variable representation. A factorization technique is presented based on a set of criteria to be satisfied by an autocorrelation function, and leading to a shaping filter with real-valued coefficients.

1.3 Summary

The state-variable representation of a system is more general than a single n-th order differential equation representation because the latter representation implies that a stringent observability condition must be satisfied by the system. In Chapter II, a state-variable model without feedback is derived for the shaping filter. The factorization problem is defined with respect to this model, and appropriate properties of the output autocorrelation function are derived.

A factorization technique valid on a finite interval is developed in Chapter III. By assuming that the autocorrelation function is sufficiently differentiable, the coefficients of the shaping filter are related to the solutions of a matrix Riccati differential equation. In order to formulate the Riccati equation, new results concerning the mean-square differentiability of a random process are developed and proved. By utilizing a set of linear constraints, the order of the Riccati equation may be reduced. If the shaping filter can be represented by a single n-th order differential equation, then the reduced Riccati equation, which is non-linear, may be transformed to a linear differential equation valid on the entire interval of interest. The order of this linear equation is the same as the order of the equation considered by Darlington.

Since the Riccati equation is non-linear, its solutions may have a finite escape time. In Chapter IV, conditions are developed which insure that a solution of the Riccati equation, and hence a shaping filter, exists globally. An important global property is stability. Stability of the shaping filter is defined in the sense that a square-integrable input produces a bounded output. If the given autocorrelation function is bounded, the shaping filter will be stable in the sense described provided that the conditions for a global solution of the Riccati equation are also satisfied. One of the most interesting results obtained provides a characterization of the autocorrelation functions which admit factorization. The characterization, related to the factorization technique employs a set of easily applied criteria. These criteria do not depend explicitly on the non-negative definite condition which, as is well-known, must be satisfied by any autocorrelation function.

In Chapter V, the class of autocorrelation functions under consideration is broadened. Cases are considered for which the Riccati equation has an isolated singular point, and for which the Riccati equation is undefined everywhere on an interval. In the former case, conditions are established which insure that the Riccati equation has a solution which is continuous at the singular point. In the latter case, the factorization problem has an algebraic solution. Autocorrelation functions in this category include those corresponding to random processes which may be represented exactly by a truncated Karhunen-Loeve expansion.

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CHAPTER II

THE STATE-VARIABLE MODEL

2.0 Introduction

This chapter introduces the definitions and mathematical limitations relevant to the treatment to follow and presents some basic results which help to define the factorization problem in a simple and concise fashion. Although many of the results discussed in this chapter are known, the proof of Theorem 2.1 and the results which conclude Section 2.3 do not seem to have appeared in the literature previously.

2.1 Representation of the Shaping Filter

The class of shaping filters to be considered includes those which may be represented by the set of linear differential equations†

$$\dot{z}(t) = A(t)\, z(t) + B(t)\, u(t) \tag{2.1a}$$

$$y(t) = C^t(t)\, z(t) \quad . \tag{2.1b}$$

The input u(t) is a real-valued r-vector (column matrix) representing a zero mean white noise process, so that

$$E\left[u(t)\, u^t(\tau)\right] = I\, \delta(t-\tau) \quad , \tag{2.2}$$

where E denotes expectation, I is an $r \times r$ identity matrix and $\delta(t-\tau)$ is the unit impulse "function". The "state" z(t) is an n-vector, and the output y(t) is a scalar. The coefficients A, B, and C are conformable, real-valued matrices of appropriate order which generally vary with time. The shaping filter corresponding to equation (2.1) is assumed to be casual, i.e., non-anticipative. A block diagram of (2.1) appears in Figure 2-1.

The class of systems represented by (2.1) generate random processes which in general, are nonstationary analogues of the random processes considered by Bode and Shannon [2.1]. That is, if the coefficients A, B, and C are constants (for all time) and if the shaping

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† The superscript t will be used to denote matrix transposition and the symbol $\dot{z}(t)$ the first derivative of z(t). When the context is clear, the explicit dependence of a function on its argument will be suppressed.

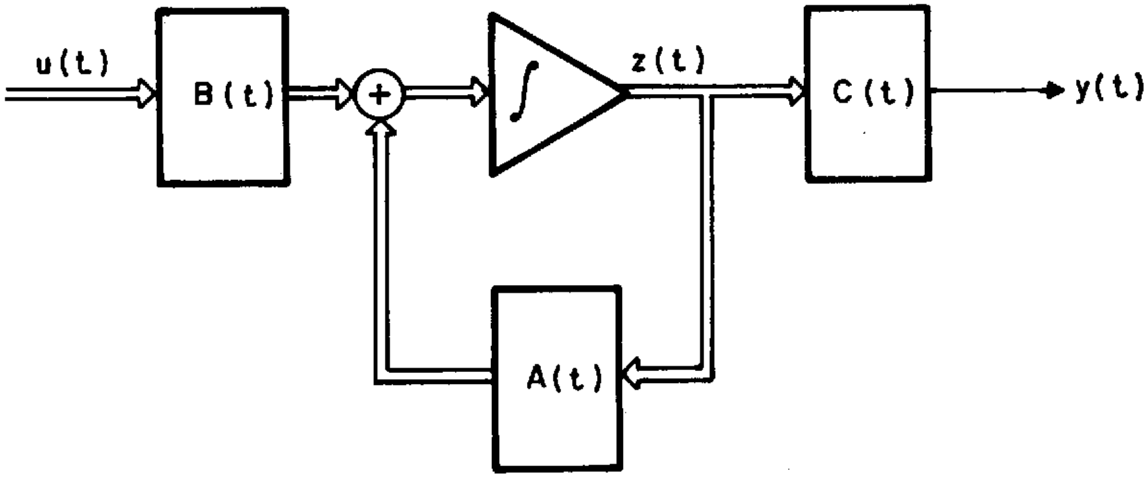

FIG. 2-1 STATE-VARIABLE MODEL OF A SHAPING FILTER. HEAVY ARROWS INDICATE VECTOR-VALUED QUANTITIES.

filter corresponding to equation (2.1) is stable, then, in the steady state, the output process y(t) is stationary and its power spectral density is a rational function of frequency.

Equation (2.1) represents a system which may be realized by the interconnection of a finite number of linear, lumped, time-variable elements. Furthermore, equation (2.1) is the most general representation of such a system.†

Despite the generality of (2.1), it is convenient to recast this equation and represent the system without feedback of the state variables. Let U(t) be a fundamental matrix solution [2.3] of equation (2.1a); that is, U(t) is an everywhere nonsingular solution of the associated homogeneous equation

$$\dot{U}(t) = A(t)\,U(t) \quad . \tag{2.3}$$

If a new state vector x(t) is defined by

$$z(t) = U(t)\,x(t) \quad , \tag{2.4}$$

the system (2.1) may easily be transformed to

$$\dot{x} = \beta\, u \tag{2.5a}$$

$$y = \phi^t\, x \quad , \tag{2.5b}$$

where the matrix β is given by

$$\beta(t) = U^{-1}(t)\,B(t) \quad , \tag{2.6}$$

† The case where there is a direct connection from input to output is excluded from consideration here since techniques somewhat simpler than those developed in the present work may be used to accommodate the case of a direct connection [2.2].

and the vector $\phi(t)$, by

$$\phi^t(t) = C^t(t)\ U(t) \quad . \tag{2.7}$$

A block diagram of (2.5) appears in Figure 2-2. If $z(t_o)$ represents an initial value of the state variable in equation (2.1) then the response of the shaping filter to an input u(t) may be written as

$$y(t) = C^t(t)\,U(t)\,U^{-1}(t_o)\,z(t_o) + \int_{t_o}^{t} C^t(t)\,U(t)\,U^{-1}(\tau)B(\tau)u(\tau)d\tau \tag{2.8}$$

or, in terms of the quantities in (2.5), as

$$y(t) = \phi^t(t)\,x(t_o) + \int_{t_o}^{t} \phi^t(t)\,\beta(\tau)\,u(\tau)\;d\tau \tag{2.9}$$

The representations of the shaping filter in equations (2.1) and (2.5) are equivalent, as is well known [2.4]. The shaping filter has the impulse response

$$h(t,\tau) = \begin{cases} \phi^t(t)\ \beta(\tau) & \text{for } t \geq \tau \\ 0 & \text{for } t < \tau \end{cases} \tag{2.10}$$

as is evident from (2.9). The function $h(t,\tau)$ is an r-vector, where r is the dimension of the input u(t).

Although the shaping filter representation in (2.5) is general and mathematically convenient, it is often unsuitable for practical simulation. For example, asymptotic stability of the state variable is not generally preserved by transformation (2.4). However, the theory of equivalent systems is sufficiently well developed to indicate when the system represented by equation (2.5) has an equivalent but practical realization [2.5, 2.6].

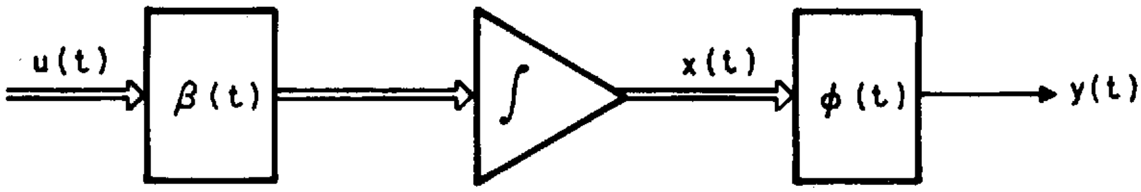

FIG. 2-2 STATE-VARIABLE MODEL OF A SHAPING FILTER WITH FEEDBACK ABSENT. HEAVY ARROWS INDICATE VECTOR-VALUED QUANTITIES

2.2 The Factorization Problem

In what follows, it should be assumed that the initial value of the state variable $x_o = x(t_o)$ is derived from a zero-mean random variable uncorrelated with the white noise input $u(t)$. The output $y(t)$ is then a real-valued, zero-mean random process with autocorrelation function

$$r(t,\tau) = E[y(t)\,y(\tau)] \quad . \tag{2.11}$$

For purposes of analysis, one may calculate $r(t,\tau)$ in terms of the coefficients $\phi(t)$ and $\beta(t)$ in a straightforward way [2.7]. If (2.9) is written as

$$y(t) = \phi^t(t)\,x_o + \int_{t_o}^{t} h(t,\tau)\,u(\tau)\,d\tau \quad , \tag{2.9a}$$

then $r(t,\tau)$ may be expressed as

$$r(t,\tau) = E\left[\left(\phi^t(\tau)\,x_o + \int_{t_o}^{t} h(t,\lambda)\,u(\lambda)d\lambda\right)\left(\phi^t(\tau)x_o + \int_{t_o}^{\tau} h(\tau,\xi)u(\xi)d\xi\right)^t\right]. \tag{2.12}$$

Since, by assumption,

$$E[u(t)\,x_o] = 0 \quad , \tag{2.13}$$

the autocorrelation function in (2.12) becomes

$$r(t,\tau) = \phi^T(t)\,E[x_o x_o^t]\phi(t) + \int_{t_o}^{t}\int_{t_o}^{\tau} h(t,\lambda)\,h^t(\tau,\xi)\,\delta(\lambda-\xi)\,d\xi\,d\lambda \tag{2.14}$$

where the impulse function in the integrand results from (2.2).

Assume that for the moment that $t > \tau$. Performing the integration with respect to the variable ξ in (2,14) yields

$$r(t,\tau) = \phi^t(t)\,E[x_o x_o^t]\,\phi(\tau) + \int_{t_o}^{t} h(t,\lambda)h^t(\tau,\lambda)\,d\lambda \quad .$$

However, the shaping filter is assumed to be causal, which implies that $h(\tau,\lambda) = 0$ for all $\lambda > \tau$. Therefore the upper limit t of the above integral may be replaced by τ.

In terms of a matrix $M(t)$ defined by

$$M(t) = E[x_o x_o^t] + \int_{t_o}^{t} \beta(\lambda)\beta^t(\lambda)\, d\lambda , \tag{2.15}$$

the autocorrelation function is

$$r(t,\tau) = \phi^t(t)\, M(\tau)\, \phi(\tau) , \tag{2.16}$$

for $t > \tau$. For $\tau > t$, the above development may be repeated to yield

$$r(t,\tau) = \phi^t(t)\, M(t)\, \phi(\tau) . \tag{2.17}$$

Thus, combining (2.16) and (2.17), we have

$$r(t,\tau) = \phi^t(t)\, M[\min(t,\tau)]\, \phi(\tau) . \tag{2.18}$$

One may easily verify that the matrix $M(t)$ is the covariance matrix of the state vector, i.e.,

$$M[\min(t,\tau)] = E[x(t)\, x^t(\tau)] , \tag{2.19}$$

where $x(t)$ is the state vector of the shaping filter. The salient properties of $M(t)$ are stated in the following Lemma, which generalizes a result of Doob [2.8].

Lemma 2.1. Let a covariance matrix M be defined as in equation (2.19). Then M is symmetric and

(a) $M(t) \geq 0$, for all t

(b) $M(t_2) - M(t_1) \geq 0$, for all $t_2 \geq t_1$. †

† For real symmetric matrices A and B, the matrix inequality $A \geq B$ means that the matrix $(A-B)$ is non-negative definite.

Proof: Symmetry of M(t) follows by equating t and τ in equation (2.19).

To prove (a), let v be an arbitrary real-valued n-vector and note that $v^t x(t)$ is a real scalar-valued random process. Then

$$v^t M(t) v = v^t E[x(t) x^t(t)]v = E[(v^t x(t))^2] \geq 0 \quad .$$

Part (b) is established as follows.

$$0 \leq E[(x(t_2) - x(t_1))(x(t_2) - x(t_1))^t)$$

$$= E[x(t_1) x^t(t_1)] + E[x(t_2) x^t(t_2)] - E[x(t_1) x^t(t_2)] - E[x(t_2) x^t(t_1)]$$

$$= M(t_2) - M(t_1).$$

The last equality follows from (2.19).

Note that M(t) as defined by equation (2.15) is differentiable, so that from the previous Lemma, $\dot{M}(t) \geq 0$.

Definition 2.1. A symmetric, differentiable, real-valued matrix M(t) will be called admissible if M(t) is non-negative definite, and non-decreasing.

The development above makes clear that admissible matrices will play a crucial role in what follows. According to (2.18), the function $r(t,\tau)$ bears a simple relation to the state variance matrix M(t). This simple relation is exploited in the following Theorem, in which several important properties of $r(t,\tau)$ are derived.

<u>Theorem 2.1.</u> Let the relation

$$r(t,\tau) = \phi^t(t)\, M[\min(t,\tau)]\, \phi(\tau) \quad ,$$

where M(t) is an admissible matrix, define a function $r(t,\tau)$. Then $r(t,\tau)$ satisfies the following:

A1. $r(t,\tau)$ is separable; i.e., there exist real-valued vectors $\phi(t)$ and $\gamma(t)$ such that

$$r(t,\tau) = \begin{cases} \phi^t(t)\,\gamma(\tau), & \text{for } t \geq \tau \\ \phi^t(\tau)\,\gamma(t), & \text{for } t < \tau , \end{cases} \tag{2.20}$$

A2. $r(t,\tau)$ is symmetric; i.e.,

$$r(t,\tau) = r(\tau,t) \quad ,$$

and

A3. $r(t,\tau)$ is non-negative definite; i.e., for any choice of instants $t_1 \leq t_2 \leq \ldots t_m$, for any choice of scalars $\alpha_1, \alpha_2, \ldots \alpha_m$, and for any finite integer m, the following quadratic form is non-negative:

$$Q = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i\, r(t_i, t_j)\, \alpha_j \geq 0 \ . \tag{2.21}$$

<u>Proof:</u> The first assertion follows by equating

$$\gamma = M\,\phi \quad . \tag{2.22}$$

Symmetry is apparent by inspection of (2.18). In order to prove the third assertion, define matrices $\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2, \ldots \boldsymbol{\Delta}_m$ as

$$\boldsymbol{\Delta}_1 = M(t_1) \quad ,$$

$$\boldsymbol{\Delta}_k = M(t_k) - M(t_{k-1}), \quad \text{for } k = 2, \ldots m \quad .$$

Then the quadratic form Q becomes

$$Q = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \, r(t_i, t_j) \, \alpha_j = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \, \phi^t(t_i) \, M[\min(t_i, t_j)] \, \phi(t_j) \, \alpha_j$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \, \phi^t(t_i) \, \boldsymbol{\Delta}_1 \, \phi(t_j) \, \alpha_j + \sum_{i=2}^{m} \sum_{j=2}^{m} \alpha_i \, \phi^t(t_i) \, \boldsymbol{\Delta}_2 \, \phi(t_j) \, \alpha_j + \ldots + \alpha_m \, \phi^t(t_m) \, \boldsymbol{\Delta}_m \; .$$

A typical term in the above sequence of summations is

$$\sum_{i=k}^{m} \sum_{j=k}^{m} \alpha_i \, \phi^t(t_i) \, \boldsymbol{\Delta}_k \, \phi(t_j) \, \alpha_j \quad . \tag{2.23}$$

Since $M(t)$ is admissible by hypothesis, each of the matrices $\boldsymbol{\Delta}_k$, for $k = 1, 2, \ldots m$, is non-negative definite. Therefore, as is well known [2.9], $\boldsymbol{\Delta}_k$ possesses a generally non-unique, real-valued square-root matrix $\boldsymbol{\Delta}_k^{\frac{1}{2}}$ so that $\boldsymbol{\Delta}_k^{\frac{1}{2}} \boldsymbol{\Delta}_k^{\frac{1}{2}t} = \boldsymbol{\Delta}_k$. Let $\phi_k(t_i) = \boldsymbol{\Delta}_k^{\frac{1}{2}} \phi(t_i)$. In terms of ϕ_k, equation (2.23) becomes

$$\sum_{i=k}^{m} \sum_{j=k}^{m} \alpha_i \, \phi_k^t(t_i) \, \phi_k(t_j) \, \alpha_j = \left\| \sum_{i=k}^{m} \alpha_i \, \phi_k(t_i) \right\|^2 \geq 0 \quad ,$$

where $\| \cdot \|$ denotes the Euclidean norm. Therefore, the quadratic form Q is a sum of non-negative quantities and is hence itself non-negative.

The first assertion of the above Theorem reflects the fact that the shaping filter is a linear system having a finite dimensional state space.

The second and third assertions indicate that $r(t,\tau)$ is an autocorrelation function. Indeed, it is well known that an arbitrary function $r(t,\tau)$ is an autocorrelation function if and only if $r(t,\tau)$ satisfies A2, and A3 of Theorem 2.1 [2.10].

The following Corollary, an important consequence of Theorem 2.1, is the well-known Schwarz inequality for random variables.

Corollary 2.1. Let $y(t)$ and $y(\tau)$ be real scalar-valued random variables, and let $r(t,\tau) = E[y(t)\,y(\tau)]$. Then

$$[r(t,\tau)]^2 \leq r(t,t)\,r(\tau,\tau) \quad .$$

Proof. Let $m=2$, $t_1 = t$, and $t_2 = \tau$ in assertion A3 of Theorem 2.1.

In the preceding discussion, a model of a shaping filter was defined and some statistical properties of its state vector and output were derived. The interesting and important problem is to proceed the opposite way; i.e., given an autocorrelation function $r(t,\tau)$, determine the quantities x_o, $\phi(t)$ and $\beta(t)$ which define the shaping filter model of (2.5). The determination of these quantities will provide a solution to the so-called factorization problem. Factorization may be defined formally as follows:

Definition 2.2. A function $r(t,\tau)$ satisfying conditions A1, and A2 of Theorem 2.1 admits a factorization if there exists a random process $y(t)$ such that

$$r(t,\tau) = E[y(t)\,y(\tau)] \quad ,$$

where $y(t)$ is generated at the output of shaping filter modelled by (2.5).

The following Theorem is now relevant, since it summarizes some of the main points of this section. It is similar to a Theorem stated by Kalman [2.7].

Theorem 2.2. A function $r(t,\tau)$ satisfying conditions A1, and A2 of Theorem 2.1 admits a factorization if and only if there is a vector $\phi(t)$ and an admissible matrix $M(t)$ such that

$$r(t,\tau) = \phi^t(t)\, M[\min(t,\tau)]\, \phi(\tau) \quad ,$$

in which case $r(t,\tau)$ also satisfies condition A3.

Proof: The "only if" part of the theorem follows directly from Lemma 2.1 and Theorem 2.1.

To prove the "if" part of the theorem, note that since $M(t)$ is assumed to be admissible, $\dot{M}(t) \geq 0$. Therefore, a real-valued square-root matrix $\beta(t)$ exists so that

$$\beta(t)\, \beta^t(t) = \dot{M}(t) \quad .$$

Hence the coefficients of the shaping filter are determined. If $t = t_0$ is chosen as the initial time, then a random initial value of the state x_0 may be chosen from an ensemble for which

$$E[x_0 x_0^t] = M_0 = M(t_0)$$

If the rank of M_0 is at most unity, then it may be possible to assign a deterministic value to x_0.

Although the above specification of the matrix $\beta(t)$ may not be unique, a shaping filter defined in terms of this quantity will generate a random process $y(t)$ possessing the specified autocorrelation function.

Implicit in Theorem 2.2 is the fact that if $r(t,\tau)$ can be shown, by any means, to admit factorization, then from Theorem 2.1, $r(t,\tau)$ must be an autocorrelation function. This consideration will be explored further in Chapter IV.

In order to proceed with the factorization problem, it is assumed that a function $r(t,\tau)$ is given satisfying the conditions:

A1. $r(t,\tau)$ is separable

A2. $r(t,\tau)$ is symmetric, and

A3. $r(t,\tau)$ is non-negative definite.

It will suffice to restrict attention to the case for which $t \geq \tau$. Then, from A1, there exist functions $\phi(t)$ and $\gamma(t)$ such that

$$r(t,\tau) = \phi^t(t)\, \gamma(\tau) \quad , \quad \text{for } t \geq \tau \quad . \tag{2.24}$$

The functions ϕ and γ are vector-valued with dimension n, where n determines the order of the shaping filter model in (2.5).

If $r(t,\tau)$ is to admit factorization, then $r(t,\tau)$ must satisfy (2.18), so that

$$\phi^t(t)\, \gamma(\tau) = \phi^t(t)\, M(\tau)\, \phi(\tau) \quad . \tag{2.25}$$

Assume that the n components $\phi_1(t), \ldots, \phi_n(t)$ are linearly independent on T, the interval of interest, and assume a similar condition for the components of the vector γ.[†] If such were not the case, then ϕ and γ might be replaced by vector-valued functions of lower dimension such that equation (2.24) remains valid. In view of the assumed linear independence of the components of $\phi(t)$ in particular, the t and τ variations in (2.25) may be equated with the result that γ and ϕ are related by

$$\gamma(t) = M(t)\,\phi(t) \quad . \tag{2.26}$$

Equation (2.26) may be regarded as the basic equation to which an admissible matrix solution M(t) must be sought in order to solve the factorization problem and obtain the desired shaping filter.

Before investigating equation (2.26) in generality in Chapter III, we will consider a suggestive special case in the next section.

[†] Unless otherwise specified, T represents a finite open interval.

2.3 First-Order Shaping Filter

A first-order shaping filter is defined by requiring the given functions $\phi(t)$ and $\gamma(t)$ to be scalar-valued. The coefficient $\beta(t)$ will then be scalar-valued. This case is important primarily because its simplicity lends insight to the general case. Although the first-order case is relevant to the factorization problem, a detailed study of this case actually appears in the classic paper of J. Mercer [2.11] which treats the theory of integral equations. The development below is based, in part, on Mercer's work.

For a first-order shaping filter the autocorrelation function satisfies

$$r(t,\tau) = \begin{cases} \phi(t)\,\gamma(\tau), & \text{for } t \geq \tau \\ \phi(\tau)\,\gamma(t), & \text{for } t < \tau \end{cases}, \tag{2.27}$$

where $\phi(t)$ and $\gamma(t)$ are now scalar-valued and the variables t and τ are allowed to vary over an open interval (a,b), where a or b, or both, may be infinite. Denote by Σ the set of points in (a,b) for which neither $\phi(t)$ nor $\gamma(t)$ vanish. Then a scalar valued function M(t) may be defined as

$$M(t) = \frac{\gamma(t)}{\phi(t)}$$

on the set Σ.

In terms of M(t), the autocorrelation function may be expressed as

$$r(t,\tau) = \phi(t)\, M\left[\min(t,\tau)\right] \phi(\tau) , \tag{2.28}$$

which is the scalar version of (2.18).

Since $r(t,\tau)$ is an autocorrelation function, $r(t,\tau)$ must satisfy condition A3. In particular, $r(t,t)$ must be non-negative, i.e.,

$$r(t,t) \geq 0 \tag{2.29}$$

for t in (a,b), and $r(t,\tau)$ must satisfy the Schwarz inequality

$$r(t,t)\, r(\tau,\tau) - r^2(t,\tau) \geq 0 \tag{2.30}$$

for t and τ in (a,b). Equation (2.29) implies that

$$M(t) = \frac{r(t,t)}{\phi^2(t)} \geq 0 \quad , \tag{2.31}$$

while (2.30) implies that

$$r(t,t)\, r(\tau,\tau) - r^2(t,\tau) = [\phi(t)\,\phi(\tau)]^2\, M(\tau)\,[M(t) - M(\tau)] \geq 0 \quad ,$$

for $t \geq \tau$. Therefore M(t) is non-decreasing and non-negative on the set Σ. If in addition M(t) is differentiable, then M(t) is admissible on Σ.

The following discussion, based on Mercer's work [2.11], establishes some preliminary results which will allow the domain of definition of M(t) to be extended to points outside of the set Σ.

Consider the points at which either (or both) of the functions $\phi(t)$ and $\gamma(t)$ becomes zero, i.e., suppose that $r(\tau,\tau) = 0$ at some point τ in (a,b). Then with t an arbitrary point in (a,b), equation (2.30) becomes

$$r(t,t)\, r(\tau,\tau) - r^2(t,\tau) = -r^2(t,\tau) \geq 0 \quad .$$

Thus

$$r(t,\tau) = r(\tau,t) = 0 \tag{2.32}$$

for all t in (a,b) if $r(\tau,\tau) = 0$ for some τ in (a,b). Equation (2.32) implies that either $\phi(\tau)$ or $\gamma(\tau)$ or both become zero. We will show that the location of the point τ in (a,b) determines which of the functions $\phi(\tau)$ or $\gamma(\tau)$ vanishes.

Let $\sigma_0 = \inf \Sigma$, and $s = \sup \Sigma$, and consider the intervals (a,σ_0), (σ_0, s), and (s,b) where $a \le \sigma_0 < s \le b$, as illustrated in Figure 2-3. Suppose $r(\tau,\tau) = 0$ and τ is in (a,σ_0). Clearly, there is a point t in Σ such that $t > \tau$. But (2.27) and (2.32) imply that

$$r(t,\tau) = \phi(t)\,\gamma(\tau) = 0 \quad .$$

Since $\phi(t)$ does not vanish for t in Σ, then $\gamma(\tau) = 0$ for τ in (a,σ_0).

Now, suppose that $r(\tau,\tau) = 0$ where τ is in (s,b). Then there is a point t in Σ such that $t < \tau$. Equations (2.27) and (2.32) imply that

$$r(t,\tau) = \phi(\tau)\,\gamma(t) = 0 \quad .$$

But $\gamma(t)$ does not vanish for t in Σ; hence $\phi(\tau) = 0$ for τ in (s,b).

If $r(\tau,\tau) = 0$ and τ is in (σ_0, s), then there are points t in Σ both to the left and to the right of τ. The arguments previously employed imply that both $\phi(\tau) = 0$ and $\gamma(\tau) = 0$ if τ is in (σ_0, s). Note that such points τ are not in Σ.

Using a development beginning with Mercer's results above, we will now show that the domain of the function $M(t)$ may be extended to

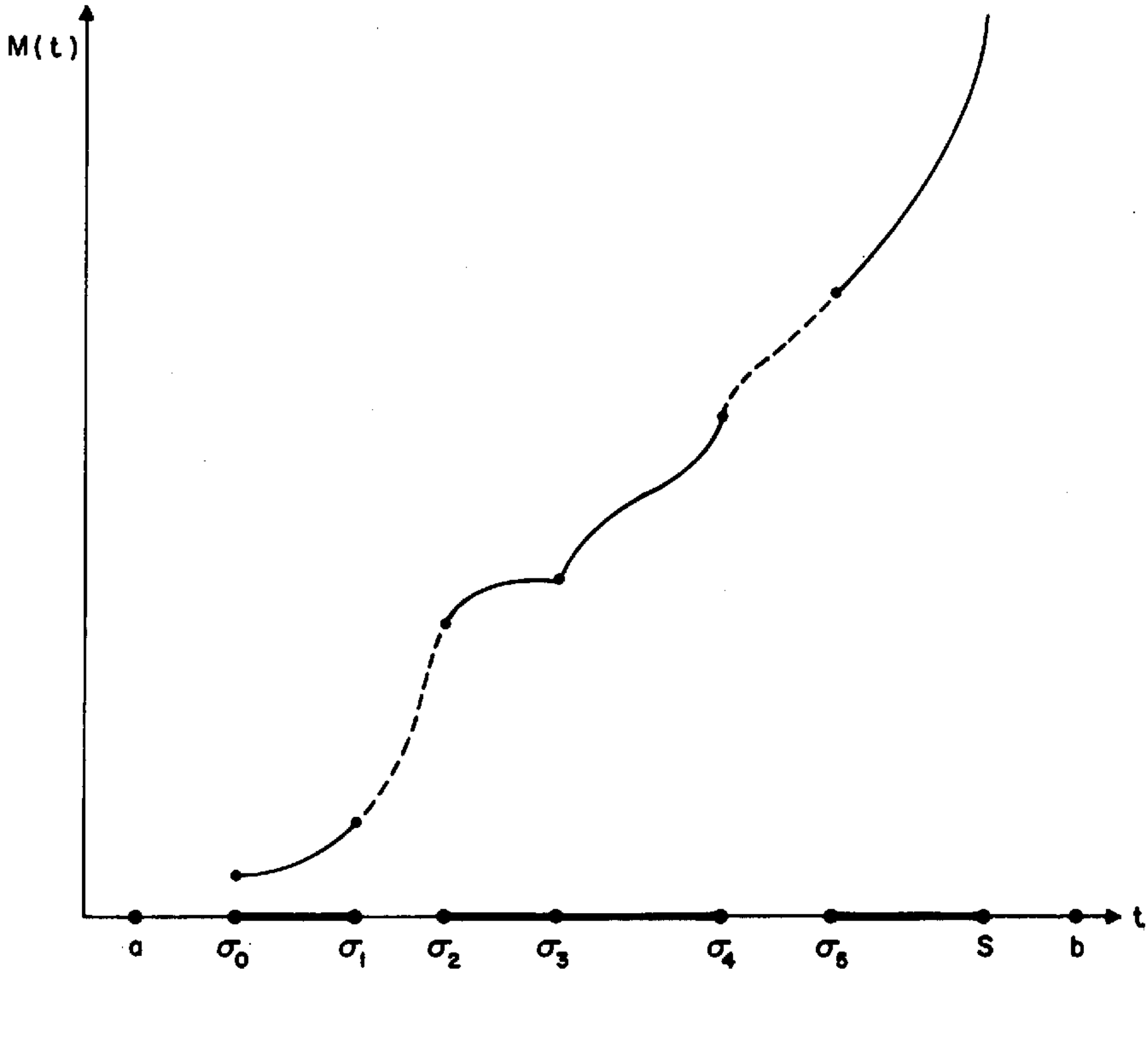

FIG. 2-3 ILLUSTRATING TYPICAL BEHAVIOR OF THE COVARIANCE M(t) DEFINED ON THE SET Σ (SOLID CURVE), AND A NON-UNIQUE EXTENSION TO POINTS NOT IN Σ (DASHED CURVE). Σ IS DENOTED BY HEAVY LINES, AND BOUNDARY POINTS OF Σ BY σ_0, σ_1, ... S.

include the half-open interval $[\sigma_0, s)$. The functions $\phi(t)$ and $\gamma(t)$ are now assumed to be continuous on (a,b). We show first that the domain of $M(t)$ may be extended to include the points of $\overline{\Sigma}$, the set of all points of Σ, and all boundary points of Σ. The results which follow do not seem to have appeared in the literature previously.

The boundary points of Σ may be ordered as

$$\sigma_0 < \cdots < \sigma_i < \cdots < s \quad ,$$

where σ_0 and s, defined previously, are also boundary points of Σ. Since s is a boundary point of Σ, then for any $\epsilon > 0$, there exists a point λ in Σ such that $|s-\lambda| < \epsilon$. Also, since neither $\phi(\lambda)$ nor $\gamma(\lambda)$ vanishes, because λ is in Σ, the ratio

$$M(\lambda) = \frac{\gamma(\lambda)}{\phi(\lambda)}$$

is finite. Moreover, from (2.31), $M(t)$ is surely non-negative for t in (a,b). Hence, if $t \leq \lambda$ and t is in Σ, then

$$0 \leq M(t) \leq M(\lambda) \quad .$$

Therefore, if σ_i is any boundary point of Σ such that $\sigma_i < \lambda$, then

$$\lim_{t \to \sigma_i} M(t) \tag{2.33}$$

exists, since $M(t)$ is bounded and monotone on Σ. But λ may be chosen arbitrarily close to s, so that $M(t)$ is defined on the set $\overline{\Sigma} - \{s\}$. Note that for some points σ_i, it may be appropriate to consider (2.33) as both a right and a left-hand limit, as would be the case if σ_i were both a right-hand and left-hand boundary point of Σ. In Figure 2-3, σ_3 is such a boundary point. Although both limits (2.33) will exist, there is no assurance that the right-hand and left-hand limits will be equal, and $M(t)$ may have a simple discontinuity at σ_i. An analogous phenomenon is possible in the n-th order case, and this will be investigated in Section 5.1.

In order to complete the extension of the domain of $M(t)$ to the interval $[\sigma_0, s)$, we must consider points t of $[\sigma_0, s)$ which are not in $\overline{\Sigma}$, but for which $r(t,t) = 0$. Clearly, t is not an isolated point for that would imply that t is a boundary point of Σ. Therefore, t must lie in a closed interval $[\sigma_i, \sigma_{i+1}]$, where σ_i and σ_{i+1} are boundary points of Σ, and $r(t,t) = 0$ for all t in $[\sigma_i, \sigma_{i+1}]$. But t is in (σ_0, s), so that the previous discussion implies that $\phi(t) = \gamma(t) = 0$. Hence, an arbitrary value may be assigned to $M(t)$ for all t in $[\sigma_i, \sigma_{i+1}]$ without violating (2.28). The argument above may be applied to all such intervals $[\sigma_i, \sigma_{i+1}]$ in order to extend the domain of $M(t)$ to include the entire interval $[\sigma_0, s)$. If $M(t)$ is admissible for t in Σ, and if $M(t)$ exists at all limit points of Σ except (perhaps) s, then the extension of $M(t)$ may be chosen to be admissible on $[\sigma_0, s)$. The dotted curves in Figure 2-3 illustrate possible extensions of $M(t)$.

If $M(t)$ is admissible then $\dot{M}(t) \geq 0$. The input multiplier $\beta(t)$ is now a scalar to be evaluated as

$$\beta(t) = \pm [\dot{M}(t)]^{\frac{1}{2}} \ ;$$

Thus a shaping filter model is determined for all t in $[\sigma_0, s)$. An initial time for generating the random process may be any point t_0 in $[\sigma_0, s)$. An initial value of the state variable x_0 may be chosen in two ways in the first-order case. Either a deterministic value may be chosen such that

$$x_0^2 = M(t_0) \quad ,$$

or x_0 may be considered a random variable with covariance

$$E[x_0^2] = M(t_0) \quad .$$

If the functions $\phi(t)$ and $\gamma(t)$ happen to be proportional, then $M(t)$ is a constant, and $\beta(t) = 0$. In this case the shaping filter is autonomous, that is, it has no input except the initial value x_0 which may be deterministic, and the random process $y(t)$, given by

$$y(t) = x_0\, \phi(t) \quad ,$$

may be deterministic. This phenomenon also occurs in the n-th order case and will be discussed further in Section 5.2.

If the points s and b coincide, the above analysis indicates that a shaping filter may be realized on $[\sigma_0, b)$. If b is infinite then the realization of the shaping filter is global. If the points s and b do not coincide, the existence of a shaping filter on (s, b) cannot be guaranteed. This matter will be explored for the n-th order case in Chapter IV.

The following example concludes this chapter. This example is not intended to display practical utility, but is chosen to illustrate in a simple case the extension of the function $M(t)$.

Example 2.1. Let

$$r(t,\tau) = \tau \cos \tau \cos t, \quad \text{for } t > \tau \quad .$$

Then $\phi(t) = \cos t$, and $\gamma(t) = t \cos t$. The interval of interest is $(0,\infty)$. The set Σ consists of all points in $(0,\infty)$ except for $t = (2n-1)\pi/2$, $n = 1, 2, \ldots$. The function $M(t)$ may be evaluated as

$$M(t) = \frac{\gamma(t)}{\phi(t)} = \frac{t \cos t}{\cos t} = t$$

for t in Σ. The points $t = (2n-1)\pi/2$ are boundary points of Σ. Since the functions $\phi(t)$ and $\gamma(t)$ are everywhere continuous, the previous discussion implies $M(t) = t$ everywhere on $(0,\infty)$. Clearly, $M(t)$ is admissible, and hence, $r(t,\tau)$ is an autocorrelation function. The following equations describe the shaping filter.

$$\dot{x}(t) = \pm\, \dot{u}(t)$$

$$y(t) = x(t) \cos t$$

The initial condition x_o must satisfy

$$E\,[x_o^2] = t_o \quad .$$

If the initial time is chosen as $t_o = 0$ then the initial condition becomes $x_o = 0$ with probability one.

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CHAPTER III†

REALIZATION OF THE SHAPING FILTER: LOCAL EXISTENCE

3.0 Introduction

In the previous chapter, basic equation (2.26) was derived and it was shown that a solution of this equation determines the coefficients of a shaping filter. The object of the present chapter is to present a general technique for solving equation (2.26) and to investigate in detail the conditions for existence of a real-valued solution. One of the main additional assumptions to be introduced in this chapter is that the input to the shaping filter is scalar valued, that is, attention will be given to a single input (and single output) shaping filter. This assumption is introduced for several important reasons. First, for purposes of simulation by analogue or other means, the requirement of a single white-noise generator is simpler and less costly than the requirement of many such generators. Second and perhaps most important, the formulation of many problems involving detection or filtering of signals is greatly facilitated by introducing not only a shaping filter, but also a whitening filter, i.e., that system which performs an operation inverse to that performed by the

† Some of the results presented in this chapter have appeared earlier in a paper by the author and H.E. Meadows [3.1].

shaping filter [3.2, 3.3, 3.4]. The whitening filter accepts at its input a possibly nonstationary random process y(t) with a specified autocorrelation function and produces at its output a stationary white-noise process u(t).

Figure 3-1 illustrates a typical application of a whitening filter in the problem of optimum prediction of a random process y(t). The whitening filter converts y(t) into white-noise u(t), and the signal processor converts u(t) into $\hat{y}(t+\alpha)$, the minimum mean-square error estimate of the predicted value of y(t). Evidently, from Figure 3-1, the signal processor may itself be regarded as a shaping filter, so that the optimum predictor is a tandem connection of a whitening filter and a shaping filter. In order fully to exploit the simplicity of the predictor model, the whitening filter should have a single output; i.e., the white-noise process u(t) should be scalar valued.

As will be shown, the factorization technique developed in this chapter is applicable to a very large class of autocorrelation functions. For practical purposes, the assumption of a single input to the shaping filter is therefore not restrictive.

If the input u(t) is scalar valued, the coefficient matrix $\beta(t)$ becomes an n-vector. Then, according to (2.15), the basic equation (2.26) may be regarded as n non-linear Volterra integral equations in which the coefficients $\beta_1(t), \ldots \beta_n(t)$ appear as unknowns. This approach was explored by Stear [3.5]. Although the scope of Stear's

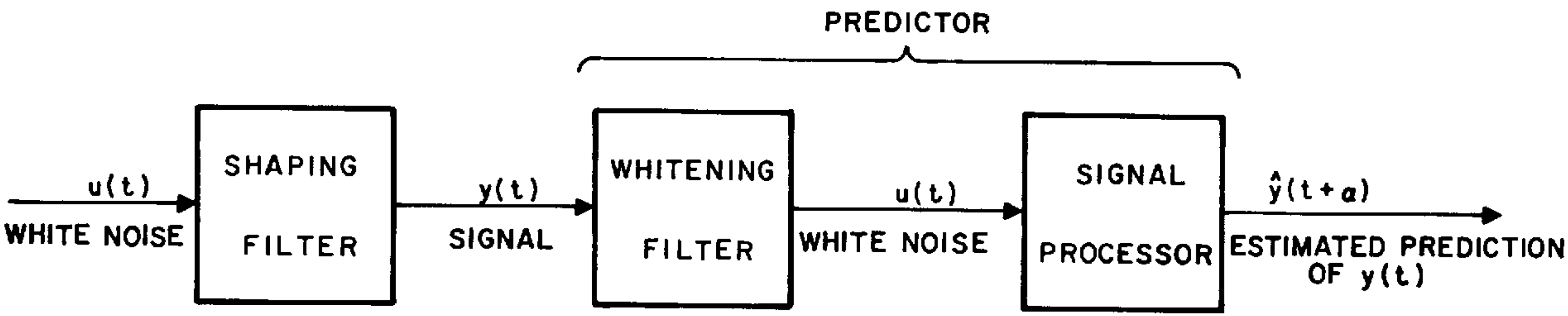

FIG. 3-1 AN OPTIMUM PREDICTOR WHICH EMPLOYS A WHITENING FILTER

work is limited in the sense that conditions for the existence of real-valued solutions to the integral equations are not related to relevant properties of the given autocorrelation function, his investigation does demonstrate an approach for realizing a time-varying shaping filter represented by (2 5).

A different approach presented in this chapter has proved quite fruitful. It will be shown that under certain appropriate and general conditions on the autocorrelation function, the basic equation may be converted to a matrix Riccati differential equation. The solution of this differential equation is the state covariance matrix $M(t)$, and real-valued coefficients of the shaping filter $\beta(t)$ may be obtained algebraically from $M(t)$.

3.1 The Derivative of a Random Process

Some new results concerning the existence of a derivative of a random process will be presented in this section. These results are pertinent to the factorization problem because a parameter defined shortly associated with differentiability of the process is used in determining the coefficients of the shaping filter.

The concepts of mean-square convergence and mean-square differentiation are defined below.

Definition 3.1. A sequence of random variables y_n converges in mean-square to a random variable y if and only if

$$\lim_{n \to \infty} E[(y_n - y)^2] = 0$$

The above limit is sometimes written

$$\underset{n \to \infty}{\text{l.i.m.}}\ y_n = y ,$$

denoting "limit in the mean."

Definition 3.2. Let $y(t)$ be a random process for which $E[y^2(t)] < \infty$ for all t in T, an interval of interest. The process $y(t)$ has a derivative in the mean-square sense, denoted by $\dot{y}(t)$ or $y^{(1)}(t)$, at a point t in T if

$$\underset{h \to 0}{\text{l.i.m.}}\ \frac{y(t+h) - y(t)}{h} = \dot{y}(t) ,$$

where $t+h$ is in T.

For brevity, the process $\dot{y}(t)$ will simply be called the derivative of $y(t)$.

A well-known condition for the existence of a derivative of a random process is stated in the following Theorem. The proof, due to Loeve [3.6], is omitted.

Theorem 3.1. $y(t)$ has a derivative at t in T if and only if the function

$$\frac{\partial^2}{\partial t\,\partial \tau} r(t,\tau)$$

exists and is finite at the point (t,t), and $r(t,\tau) = E[y(t)\,y(\tau)]$.

Theorem 3.1 is difficult to apply, as the following example indicates.

Example 3.1. Let $r(t,\tau) = e^{-|t-\tau|}$. This is the form of the autocorrelation function of a random process $y(t)$ appearing at the output of a single time-constant, R-C low-pass filter excited by white noise.

Then

$$\frac{\partial^2}{\partial t\,\partial \tau} r(t,\tau) = -e^{-|t-\tau|} \quad ,$$

for all $t \neq \tau$. One might be tempted to assume that the above equality represents $E[\dot{y}(t)\,\dot{y}(\tau)]$; however such is not the case. As will be verified later in this section, the correct expression for the autocorrelation function of $\dot{y}(t)$ is

$$E[\dot{y}(t)\,\dot{y}(\tau)] = -e^{-|t-\tau|} + 2\delta(t-\tau) \quad ,$$

which does not exist in the classical sense. Therefore, the random process corresponding to $r(t,\tau)$ is not mean-square differentiable.

We shall show that for a random process possessing an autocorrelation function with continuous partial derivatives, a more easily applicable differentiation criterion is available.

Assume that the autocorrelation function may be written in separable form as

$$r(t,\tau) = \begin{cases} \phi^t(t)\,\gamma(\tau) & \text{for } t \geq \tau \\ \phi^t(\tau)\,\gamma(t) & \text{for } t < \tau \ , \end{cases} \tag{3.1}$$

and that the vector-valued functions $\phi(t)$ and $\gamma(t)$ are continuously differentiable. Define a function $d_0^2(t)$ as

$$d_0^2(t) = \phi^t(t)\,\dot{\gamma}(t) - \dot{\phi}^t(t)\,\gamma(t) \quad . \tag{3.2}$$

The function $d_0^2(t)$ may be regarded as the variance of a random variable, and is therefore non-negative, as the following Lemma shows.

<u>Lemma 3.1</u>. Let $y(t)$ be a random process with an autocorrelation function expressed by equation (3.1), and let $d_0^2(t)$ be defined by equation (3.2). Then

$$d_0^2(t) = \lim_{t' \to t} \frac{E[(y(t') - y(t))^2]}{t' - t} \geq 0 \ ,$$

for $t' > t$.

Proof. Consider the quadratic form

$$Q = [1 \quad -1] \begin{bmatrix} r(t,t) & r(t,t') \\ r(t',t) & r(t',t') \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \geq 0 \ ,$$

which is non-negative because $r(t,\tau)$ is non-negative definite by assumption. Then

$$Q = [1 \quad -1] \ E \left\{ \begin{bmatrix} y(t) \\ y(t') \end{bmatrix} [y(t) \quad y(t')] \right\} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= E \left\{ [1 \quad -1] \begin{bmatrix} y(t) \\ y(t') \end{bmatrix} \right\}^2 = E[(y(t') - y(t))^2]$$

Therefore, for $t' > t$,

$$\frac{Q}{t' - t} \geq 0 \ ,$$

and if the limit exists,

$$\lim_{t' \to t} \frac{Q}{t' - t} \geq 0 \ .$$

Substituting (3.1) into the definition of Q and dividing by $t' - t$ yields

$$\frac{Q}{t' - t} = \frac{\phi^t(t')\,[\gamma(t') - \gamma(t)] - [\phi(t') - \phi(t)]\,\gamma(t)}{t' - t} \ .$$

Since $\dot{\phi}(t) = \lim_{t' \to t} \frac{\phi(t') - \phi(t)}{t' - t}$ and $\dot{\gamma}(t) = \lim_{t' \to t} \frac{\gamma(t') - \gamma(t)}{t' - t}$

it is clear that

$$d_o^2(t) = \lim_{t' \to t} \frac{Q}{t' - t} < \infty \ .$$

Therefore, $d_o^2(t)$ exists and is non-negative.

According to Lemma 3.1, $d_o^2(t)$ is either positive or zero for any time t. If $d_o^2(t) = 0$ then the following Theorem applies.

Theorem 3.2. Let y(t) be a random process having an auto-correlation function expressed by equation (3.1) and let $d_o^2(t)$ be defined by (3.2). Then y(t) is differentiable at a point t if and only if $d_o^2(t) = 0$.

Proof. To prove the "only if" part, note that if $\dot{y}(t)$ exists, then

$$E[\dot{y}^2(t)] = \lim_{t' \to t} \frac{E[(y(t') - y(t))^2]}{(t' - t)^2} < \infty \quad .$$

Because of the continuity hypotheses in (3.1), the above equation may be expressed as

$$E[\dot{y}^2(t)] = \lim_{t' \to t} \frac{d_o^2(t) + \epsilon(t',t)}{t' - t}$$

where

$$\lim_{t' \to t} \epsilon(t',t) = 0 \quad .$$

Therefore the variance, $E[\dot{y}^2(t)]$, is finite only if $d_o^2(t) = 0$. To prove the "if" part we shall show that $\frac{\partial^2}{\partial t \, \partial \tau} r(t,\tau)$ exists and is finite at $\tau = t$, if $d_o^2(t) = 0$. Let t be fixed and let τ vary. Consider the continuous function $\frac{\partial}{\partial t} r(t,\tau)$. From (3.1),

$$\frac{\partial}{\partial t} r(t,\tau) = \begin{cases} \dot{\phi}^t(t)\,\gamma(\tau) & \text{for } t \geq \tau \\ \phi^t(\tau)\,\dot{\gamma}(t) & \text{for } t \leq \tau \quad . \end{cases}$$

If $d_o^2(t) = 0$, then $\frac{\partial}{\partial t} r(t,\tau)$ is a continuous function of τ at the point $\tau = t$. Since the functions $\phi(t)$ and $\gamma(t)$ are continously differentiable, the function $\frac{\partial}{\partial t} r(t,\tau)$ has both a left-hand and right-hand derivative with respect to τ at the point $\tau = t$. These derivatives are equal and their common value is

$$\lim_{\tau \to t} \frac{\partial^2}{\partial t\, \partial \tau} r(t,\tau) = \dot{\phi}^t(t)\, \dot{\gamma}(t) \quad .$$

Note that since the functions $\dot{\phi}(t)$ and $\dot{\gamma}(t)$ are assumed to be continuous, then

$$\frac{\partial^2}{\partial t\, \partial \tau} r(t,\tau) = \frac{\partial^2}{\partial \tau\, \partial t} r(t,\tau)$$

for all t and τ in T. Therefore, by Theorem 3.1, the existence of the mixed partial derivative of $r(t,\tau)$, evaluated at $\tau = t$, implies the existence of the derivative process, $\dot{y}(t)$.

In the above Theorem and previous Lemma, the convenient but unnecessary assumption was made that $r(t,\tau)$ is separable. If this assumption is removed, then $d_o^2(t)$ must be redefined as

$$d_o^2(t) = \frac{\partial}{\partial \tau} r(t,\tau) - \frac{\partial}{\partial t} r(t,\tau)\bigg|_{\tau = t} \quad , \tag{3.3}$$

where τ approaches t through values $\tau < t$. Lemma 3.1 remains valid under this extended definition of $d_o^2(t)$. If it is assumed that the function $r(t,\tau)$ is continuous and that the functions $\frac{\partial}{\partial t} r(t,\tau)$, $\frac{\partial^2}{\partial t\, \partial \tau} r(t,\tau)$, and $\frac{\partial^2}{\partial \tau\, \partial t} r(t,\tau)$ exist and are continuous for all t and τ such that $t \neq \tau$, then Theorem 3.2 remains valid, and the proof requires only trivial modification. All further results in this section will be stated assuming a separable autocorrelation function. However, the general case may be accommodated by slight modification of these results.

Theorem 3.2 has some important corollaries which will be used in the sequel. These corollaries require index functions $d_i^2(t)$ to be defined as follows.

$$d_i^2(t) = \phi^{(i)t}(t)\,\gamma^{(i+1)}(t) - \phi^{(i+1)}(t)\,\gamma^{(i)}(t)^{\dagger} \tag{3.4}$$

Corollary 3.1. Let the functions ϕ and γ possess at least k continuous derivatives. Then a random process y(t) has a k-th derivative, $y^{(k)}(t)$, at a point t if and only if the functions $d_0^2(t)$, $d_1^2(t)$, ... $d_{k-1}^2(t)$ are all zero at t. Then, if k+1 derivatives of ϕ and γ exist, thus allowing $d_k^2(t)$ to be defined, the inequality $d_k^2(t) \geq 0$ is valid at the point t. Finally, if $y^{(k)}(t)$ exists for all t, then $E[y^{(k)}(t)\, y^{(k)}(\tau)] = \frac{\partial^{2k}}{\partial t^k \partial \tau^k} r(t,\tau)$, for all t and τ.

Proof. The first and second assertions follows from the previous theorems and by induction on k. The last assertion follows from a Theorem of Loeve([3.6], p 471).

The statement of the next Corollary concerns the existence of a matrix $R_k(t)$ whose i,j-th element is defined below.

† The k-th derivative of a random process or function f(t) will be denoted by $f^{(k)}(t)$.

$$[R_k(t)]_{ij} = \frac{\partial^{i+j}}{\partial t^i \, \partial \tau^j} r(t,\tau)\Big|_{\tau=t} \quad , \tag{3.5}$$

where $0 \le i,\ j \le k$, and τ approaches t through values $\tau < t$.

Corollary 3.2. Let the functions ϕ and γ possess at least k derivatives, and let $d_i^2(t) = 0$ for $i = 0,1, \ldots k-1$. Then the matrix $R_k(t)$ is well-defined, symmetric, and non-negative definite.

Proof. From Corollary 3.1, the derivatives $y^{(i)}(t)$, for $i = 0,1, \ldots k$, exist. Therefore

$$E\left[\left(y^{(i)}(t)\right)^2\right] = \frac{\partial^{2i}}{\partial t^i \, \partial \tau^i} r(t,\tau)\Big|_{\tau=t} < \infty \quad ,$$

and from the Schwarz inequality,

$$E[y^{(i)}(t)\ y^{(j)}(t)] \le E\left[\left(y^{(i)}(t)\right)^2\right] E\left[\left(y^{(j)}(t)\right)\right] < \infty \ .$$

Hence $R_k(t)$ is well-defined.

Define a vector $Y_k(t)$ as

$$Y_k(t) = \text{col. } [y^{(o)}(t)\ y^{(1)}(t) \ldots y^{(k)}(t)] \tag{3.6}$$

The previous results imply that

$$R_k(t) = E\,[Y_k(t)\ Y_k^{\,t}(t)] \, , \tag{3.7}$$

which is obviously symmetric. Let v be an arbitrary (k+1)-vector. Then $v^t Y_k(t)$ is a scalar-valued random variable and

$$v^t R_k(t)\, v = E\,[(v^t Y_k(t))^2] \ge 0 \ ,$$

which completes the proof.

It is possible that a matrix $R_k(t)$ may be defined even if the hypotheses of Corollary 3.2 are violated, but it is generally true that if $y^{(k)}(t)$ does not exist, the matrix $R_k(t)$ will be neither symmetric nor non-negative definite. The following example illustrates this remark.

Example 3.1 (continued).

$$r(t,\tau) = e^{-|t-\tau|}$$

Let $\phi(t) = e^{-t}$ and $\gamma(t) = e^{\tau}$. Substituting these functions into equation (3.2) yields $d_0^2(t) = 2$. Therefore, the random process $y(t)$ corresponding to the specified autocorrelation function is not differentiable. The matrix $R_0(t)$ reduces to the scalar $r(t,t) = 1 > 0$. The matrix $R_1(t)$ exists and may be evaluated as

$$R_1 = \begin{bmatrix} e^{-|t-\tau|} & - e^{-|t-\tau|} \\ e^{-|t-\tau|} & - e^{-|t-\tau|} \end{bmatrix}_{\tau=t} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

which is neither symmetric nor non-negative definite.

The following Corollary, which follows immediately from the preceding ones, uses the previously established results in order to formulate conditions which must be satisfied by an autocorrelation function.

Corollary 3.3. Let ϕ and γ possess a sufficient number of continuous derivatives and let a function $r(t,\tau)$ be defined as in equation (3.1). Let $d_i^2(t) \equiv 0$ for all t and all i, $0 \leq i \leq k-1$, and let $d_k^2(t)$ be non-zero everywhere. Then, necessary conditions for $r(t,\tau)$ to be an autocorrelation function are that $d_k^2(t) > 0$ for all t and $R_k(t) \geq 0$ for all t.

It will be shown in Chapter IV that the conditions stated in the previous Corollary are sufficient as well as necessary, provided that $r(t,\tau)$ is separable.

The results obtained thus far in this chapter are applicable to both separable and non-separable autocorrelation functions. We now assume that $r(t,\tau)$ is separable and furthermore, that if $r(t,\tau)$ admits factorization, the shaping filter is realizable with a scalar-valued input. These assumptions allow the index functions $d_i^2(t)$ to be related to the coefficients of the shaping filter.

Recall the basic equation,

$$\gamma(t) = M(t)\,\phi(t) \quad , \tag{3.8a}$$

where

$$\dot{M} = \beta\,\beta^t \quad , \tag{3.8b}$$

and β is vector-valued by assumption of a single input.

The following Lemma relates the index function $d_i(t)$ to the coefficients $\beta(t)$.

<u>Lemma 3.2</u>. Let $r(t,\tau)$ be a separable autocorrelation function and let $\phi(t)$ and $\gamma(t)$ possess at least k+1 continuous derivatives. Let $d_i^2(t) \equiv 0$ everywhere for $0 \leq i \leq k-1$ and let $d_k^2(t) \neq 0$ everywhere.

Then $d_i(t) = \phi^{(i)t}(t)\,\beta(t)$ for all t and for all i, $0 \le i \le k$.

Proof. The assertion is proved by repeated differentiation of basic equation (3.8). Differentiating (3.8a) and pre-multiplying the result by ϕ^t results in

$$\phi^t \dot{\gamma} = \phi^t M \dot{\phi} + \phi^t \dot{M} \phi \quad . \tag{3.9}$$

Pre-multiplying the basic equation by $\dot{\phi}^t$ yields, since M is symmetric,

$$\dot{\phi}^t \gamma = \dot{\phi}^t M \phi = \phi^t M \dot{\phi} \ . \tag{3.10}$$

Subtracting (3.10) from (3.9) produces

$$d_0^2 = \phi^t \dot{\gamma} - \dot{\phi}^t \gamma = \phi^t \dot{M} \phi$$

$$= \phi^t \beta \beta^t \phi \quad = (\phi^t \beta)^2 = 0$$

which vanishes by hypothesis. Thus

$$\dot{\gamma} = M \dot{\phi} + \dot{M} \phi = M \dot{\phi} + \beta(\beta^t \phi)$$

or

$$\dot{\gamma} = M \phi \quad , \tag{3.11}$$

everywhere. The results of operating on (3.11) just as on (3.8) above are

$$d_1(t) = \dot{\phi}^t(t)\,\beta(t) = 0 \ ,$$

and one considers next the equation

$$\gamma^{(2)} = M \phi^{(2)}$$

etc. Continuation of the sequence of operations obviously yields

$$d_i(t) = \phi^{(i)t}(t)\ \beta(t) = 0 \qquad \text{for } 0 \le i \le k-1 \tag{3.12a}$$

and for all t. The procedure described above terminates at the next step, which yields

$$d_k^2(t) = \left(\phi^{(k)t}(t)\ \beta(t)\right)^2 > 0 \tag{3.12b}$$

where the inequality follows from Corollary 3.1.

If the impulse response of the shaping filter is denoted by $h(t,\tau)$, the result just proved may be stated as

$$d_i(t) = \frac{\partial^i}{\partial t^i} h(t,\tau)\bigg|_{\tau=t} \quad , \text{ for } 0 \le i \le k \ .$$

If $r(t,\tau)$ admits factorization, then the index function $d_k(t)$ has an interesting physical interpretation. Let the hypotheses of Lemma 3.2 hold and let the equations

$$\dot{x} = \beta\, u \tag{3.13a}$$

$$y = \phi^t x \tag{3.13b}$$

describe the shaping filter. Corollary 3.1 allows equation (3.13b) to be differentiated k times. Thus

$$y^{(1)} = \dot{\phi}^t x + \phi^t \dot{x} = \dot{\phi}^t x + (\phi^t \beta) u = \dot{\phi}^t x + d_o u$$

or

$$y^{(1)} = \dot{\phi}^t x \ , \tag{3.14}$$

and

$$y^{(2)} = \phi^{(2)t} x + d_1 u = \phi^{(2)} x \ .$$

Similarly,

$$y^{(i)} = \phi^{(i)t} x, \qquad \text{for } 0 \le i \le k \tag{3.15}$$

Proceeding formally, one may perform an additional differentiation (actually forbidden by Corollary 3.1) to obtain from (3.15) the following equations:

$$\dot{x} = \beta u \tag{3.16a}$$

$$y^{(k+1)} = \phi^{(k+1)t} x + d_k u \quad . \tag{3.16b}$$

Equation (3.16) may be regarded as a state-variable model for a shaping filter which transforms white-noise, u(t), into a process denoted by $y^{(k+1)}(t)$. Since the function $d_k(t) \neq 0$ for all t, the process $y^{(k+1)}(t)$ will contain a white-noise component for all t, and the autocorrelation function of $y^{(k+1)}(t)$ will contain an impulse. The autocorrelation of $y^{(k+1)}$ may be computed using, in part, the results of Section 2.2, and some results of Newcomb and Anderson [3.7] which treat shaping filters with a direct connection from input to output. Integrating equation (3.16a) yields

$$x(t) = x_o + \int_{t_o}^{t} \beta(\lambda)\, u(\lambda)\, d\lambda$$

Then the cross-correlation of state vector and input is

$$E[x(t)\,u(\tau)] = E\,[x_o\, u(\tau)] + \int_{t_o}^{t} \beta(\lambda)\, E[u(\lambda)\, u(\tau)]\, d\lambda$$

$$= \begin{cases} \beta(\tau) & \text{for } t > \tau \\ 0 & \text{for } t < \tau \end{cases} \tag{3.17}$$

Since $E[x(t)x^t(\tau)] = M[\min(t,\tau)]$, the autocorrelation function

$$E[y^{(k+1)}(t)\, y^{(k+1)}(\tau)] = E\Big[\Big(\phi^{(k+1)t}(t)\, x(t) + d_k(t)\, u(t)\Big) \cdot \Big(\phi^{(k+1)}(\tau)\, x(\tau) + d_k(\tau)\, u(\tau)\Big)\Big] ,$$

may be evaluated in terms of the unit step function $S(t)$ as

$$E[y^{(k+1)}(t)\, y^{(k+1)}(\tau)] = \phi^{(k+1)}(t)\, M[\min(t,\tau)]\, \phi^{(k+1)t}(\tau) + d_k^2(t)\, \delta(t-\tau) + \phi^{(k+1)t}(t)\, \beta(\tau)\, d_k(\tau)\, S(t-\tau) + \phi^{(k+1)t}(\tau)\, \beta(t)\, d_k(t)\, S(\tau - t) \quad . \tag{3.18}$$

Note that if $\phi^{(k+1)}$ and $\gamma^{(k+1)}$ exist then $\dfrac{\partial^{2(k+1)}}{\partial t^{k+1}\, \partial \tau^{k+1}}\, r(t,\tau)$ exists for all $t \neq \tau$. For $t > \tau$,

$$\frac{\partial^{2(k+1)}}{\partial t^{k+1}\, \partial \tau^{k+1}}\, r(t,\tau) = \frac{\partial^2}{\partial t\, \partial \tau}\Big[\phi^{(k)}(t)\; M(\tau)\; \phi^{(k)}(\tau)\Big]$$

$$= \phi^{(k+1)t}(t)\; M(\tau)\; \phi^{(k+1)}(\tau) + \phi^{(k+1)t}(t)\; \beta(\tau)\, d_k(\tau) \; .$$

A similar equation by be derived if $t < \tau$.

In terms of the above expression, equation (3.18) may be rewritten as

$$E[y^{(k+1)}(t)\; y^{(k+1)}(\tau)] = \frac{\partial^{2(k+1)}}{\partial t^{k+1}\, \partial \tau^{k+1}}\, r(t,\tau) + d_k^2(t)\, \delta(t-\tau) . \tag{3.19}$$

From equation (3.19), the quantity $d_k^2(t)$ may be interpreted as the "instantaneous power" associated with the white noise component of $y^{(k+1)}(t)$.

The following examples illustrating several results derived above, will conclude this section.

Example 3.1 (continued).

Let $\phi(t) = e^{-t}$ and $\gamma(t) = e^{t}$. We have shown that $d_0^2(t) = 2$. The basic equation (3.8) becomes

$$e^{t} = M(t)\, e^{-t} \quad ,$$

which has the unique solution $M(t) = e^{2t}$, $M(t)$ is admissible according to Definition 2.1, and

$$\beta(t) = [\dot{M}(t)]^{\frac{1}{2}} = \sqrt{2}\, e^{t} \quad .$$

The autocorrelation function of the process $y(t)$ may now be computed by means of (3.18) as

$$E[y(t)\, y(\tau)] = -e^{-|t-\tau|} + 2\,\delta(t-\tau)$$

The instantaneous power associated with the white-noise component of $y(t)$ is 2 watts.

Example 3.2. Let $r(t,\tau) = f(t) + f(\tau)$, where $f(t)$ is differentiable. Let

$$\phi(t) = \begin{bmatrix} f(t) \\ 1 \end{bmatrix}$$

and

$$\gamma(t) = \begin{bmatrix} 1 \\ f(t) \end{bmatrix}$$

The functions $d_o(t)$ and $d_1(t)$ may be evaluated using equation (3.4) and $d_o(t) = d_1(t) = 0$ for all t. The matrix $R_1(t)$, defined by equation (3.5) may be expressed as

$$R_1(t) = \begin{bmatrix} 2\,f(t) & \dot{f}(t) \\ \dot{f}(t) & 0 \end{bmatrix}$$

and $\det R_1(t) = -\dot{f}^2(t) < 0$. Therefore, Corollary 3.3 implies that $r(t,\tau)$, as given above, is not an autocorrelation function.

<u>Example 3.3.</u> Let $r(t,\tau) = \phi^t(t)\,\Lambda\,\phi(\tau)$, where Λ is a constant diagonal matrix for which $[\Lambda]_{ii} = \lambda_i \geq 0$. Then Λ is an admissible matrix, and from Theorem 2.1, $r(t,\tau)$ is an autocorrelation function. Assume that the function $\phi(t)$ is differentiable an arbitrary number of times. Then $d_i(t) \equiv 0$ for $i = 0, 1, \ldots$.

Define a matrix $\Phi_k(t)$ as

$$\Phi_k(t) = [\phi^{(o)}(t)\ \phi^{(1)}(t)\ \ldots\ \phi^{(k)}(t)]$$

Then R_k may be written

$$R_k = \Phi_k^t\,\Lambda\,\Phi_k\ .$$

Clearly, $R_k(t) \geq 0$ on T for $k = 0, 1, \ldots$.

This example exhibits a class of autocorrelation functions for which $d_i(t) \equiv 0$ for all $i \geq 0$. It will be shown in Chapter V that the representation $r(t,\tau) = \phi^t(t)\,\Lambda\,\phi(\tau)$ exhausts the class of autocorrelation functions for which $d_i(t) \equiv 0$ for all $i \geq 0$.

3.2. Local Solutions of the Factorization Problem

In this section, we will show that the state convariance matrix, $M(t)$, may be obtained as a solution of a matrix Riccati differential equation. The desired random process $y(t)$ is assumed to commence at a finite time and to have finite duration. Although these assumptions have practical motivation, since any experiment or simulation procedure must have finite duration, they are introduced primarily for analytical convenience. The latter assumption will be relaxed in Section 4.1.

The finite interval of interval of interest is denoted by T, and the autocorrelation function $r(t,\tau)$ is prescribed on a square $T \times T$,[†] and is assumed to satisfy conditions A1 - A5 which follow.

A1. $r(t,\tau)$ is separable

A2. $r(t,\tau)$ is symmetric

A3. $r(t,\tau)$ is non-negative definite

A4. The given functions $\phi(t)$ and $\gamma(t)$ possess at least $k+1$ continuous derivatives on T

A5. $d_i(t) \equiv 0$ on T, for $0 \le i \le k-1$

$d_k(t) \neq 0$ everywhere on T

Recall that conditions A1 - A3 were defined more completely in Theorem 2.1. The results of the previous section become valid on a finite interval by

---

[†] $T \times T$ denotes the set of points in the t,τ plane where both variables t and τ are restricted to the interval T.

replacing "everywhere" with "everywhere on T."

As a notational convenience, the following matrices are introduced.

$$\Phi_k(t) = [\phi^{(o)}(t)\ \phi^{(1)}(t)\ \ldots\ \phi^{(k)}(t)] \tag{3.20a}$$

$$\Gamma_k(t) = [\gamma^{(o)}(t)\ \gamma^{(1)}(t)\ \ldots\ \gamma^{(k)}(t)] \tag{3.20b}$$

The matrices Φ_k and Γ_k may be related by a simple expression which involves the function d_k^2. This relationship, stated in Lemma 3.3 below, will be useful in developing the main results of this section.

Lemma 3.3. Let $r(t,\tau)$ satisfy conditions A1 - A5. Denote by e_1 the (k+1)-dimensional unit vector which contains zeros in the first k positions, and unity in the last position. Then

$$\gamma^{(k+1)t}\,\Phi_k - \phi^{(k+1)t}\,\Gamma_k = e_1{}^t\, d_k^2 .$$

Proof. Consider the identity

$$\gamma^{(k)t}\,\phi^{(i)} = \phi^{(k)t}\,\gamma^{(i)}\ , \text{ for } 0 \le i \le k-1\ ,$$

which is valid for all t in T from Corollary 3.2. Differentiating this identity yields

$$\gamma^{(k+1)t}\phi^{(i)} + \gamma^{(k)t}\,\phi^{(i+1)} = \phi^{(k)t}\,\gamma^{(i+1)} + \phi^{(k+1)t}\,\gamma^{(i)}\ .$$

Corollary 3.2 implies that the terms adjacent to the equal sign are identical. Hence

$$\gamma^{(k+1)t}\phi^{(i)} = \phi^{(k+1)t}\,\gamma^{(i)}\ , \text{ for } 0 \le i \le k-1\ .$$

For i = k, the Lemma is obviously valid by definition of d_k^2.

The main results of this section concern the existence of a solution of the factorization problem. These results are stated below.[†]

Theorem 3.3. Let $r(t,\tau)$ satisfy conditions A1 - A5. Then the following assertions are valid.

(i) If $\gamma(t) = M(t)\,\phi(t)$ on T, where M(t) is symmetric and rank of $\dot{M}(t) \leq 1$, then M(t) satisfies the following Riccati differential equation:

$$\dot{M} = \frac{(\gamma^{(k+1)} - M\phi^{(k+1)})(\gamma^{(k+1)} - M\phi^{(k+1)})^t}{d_k^2} . \qquad (3.21)$$

(ii) Let M_o be any symmetric non-negative definite matrix which satisfies

$$\Gamma_k(t_o) = M_o\,\Phi_k(t_o) . \qquad (3.22)$$

If M(t) is the solution of equation (3.21) having the initial value $M(t_o) = M_o$, then M(t) is admissible and $\Gamma_k(t) = M(t)\,\Phi_k(t)$ for all t in a neighborhood of t_o.

Furthermore, the coefficient $\beta(t)$ may be evaluated as

$$\beta(t) = \frac{\gamma^{(k+1)}(t) - M(t)\,\phi^{(k+1)}(t)}{d_k(t)} \qquad (3.23)$$

† Although derived independently, the methods used to obtain the results stated in Theorem 3.3 are essentially the same as those employed by Anderson [3.8], who obtained similar results for the special case $k = 0$.

Proof. (i) The proof of the first part of the theorem follows directly from the proof Lemma 3.2, where the following equations were derived:

$$\gamma^{(k)} = M\,\phi^{(k)} \quad ,$$

and

$$d_k = \phi^{(k)}\,\beta \quad .$$

Differentiating the former equation yields

$$\gamma^{(k+1)} = M\,\phi^{(k+1)} + \dot{M}\,\phi^{(k)} \ .$$

Since M is symmetric and rank $(M) \le 1$, M may be expressed as $M = \beta\beta^t$. Then

$$\gamma^{(k+1)} = M\phi^{(k+1)} + \beta\,[\,\beta^t\phi^{(k)}\,] \quad ,$$

or

$$\beta = \frac{\gamma^{(k+1)} - M\phi^{(k+1)}}{d_k}$$

Post-multiplying this expression for β by its transpose yields the desired Riccati differential equation.

(ii) The second assertion will be proved by exhibiting a homogeneous linear differential equation, the solution of which is a vector of dimension $n(k+1)$ which has as components, the columns of the matrix $\Gamma_k - M\,\Phi_k$.

Consider the right-hand side of the following identity:

$$\frac{d}{dt}\,(\Gamma_k - M\Phi_k) = (\dot{\Gamma}_k - M\,\dot{\Phi}_k) - \dot{M}\,\Phi_k \ .$$

Substituting equation (3.21) for $\dot{M}$ yields

$$(\dot{\Gamma}_k - M\dot{\Phi}_k) - \frac{(\gamma^{(k+1)} - M\phi^{(k+1)})(\gamma^{(k+1)t}\Phi_k - \phi^{(k+1)t} M\Phi_k)}{d_k^2} .$$

If the quantity $\phi^{(k+1)t}\Gamma_k$ is added and subtracted in the right-most parentheses, the following expression results.

$$(\dot{\Gamma}_k - M\dot{\Phi}) - \frac{(\gamma^{(k+1)} - M\phi^{(k+1)})(\gamma^{(k+1)t}\Phi_k - \phi^{(k+1)t}\Gamma_k + \phi^{(k+1)t}(\Gamma_k - M\Phi_k))}{d_k^2} .$$

By invoking Lemma 3.3, this expression simplifies to

$$(\dot{\Gamma}_k - M\dot{\Phi}) - (\gamma^{(k+1)} - M\phi^{(k+1)})\left[e_1^t + \frac{\phi^{(k+1)t}(\Gamma_k - M\Phi_k)}{d_k^2}\right] ,$$

which may be expanded as

$$\frac{d}{dt}(\Gamma_k - M\Phi_k) = [\dot{\Gamma}_k - M\dot{\Phi}_k - (\gamma^{(k+1)} - M\phi^{(k+1)})e_1{}^t]$$
$$- \frac{(\gamma^{(k+1)} - M\phi^{(k+1)})\phi^{(k+1)t}}{d_k^2}(\Gamma_k - M\Phi_k). \qquad (3.24)$$

Define a matrix A as

$$A = -\frac{(\gamma^{(k+1)} - M\phi^{(k+1)})\phi^{(k+1)t}}{d_k^2} ,$$

and let q_i denote the i-th column of the matrix $\Gamma_k - M\Phi_k$. From the standard existence theorem for ordinary differential equations [3.9], the matrix A defined in terms of M exists only in a neighborhood of the point $t = t_0$. In terms of A and q_i, (3.24) may be rewritten as the following set of differential equations.

$$
\begin{aligned}
\dot{q}_0 &= Aq_0 + q_1 \\
\dot{q}_1 &= Aq_1 + q_2 \\
&\vdots \\
\dot{q}_i &= Aq_i + q_{i+1} \\
&\vdots \\
\dot{q}_k &= Aq_k
\end{aligned}
\tag{3-25}
$$

Equation (3.25) may be abbreviated further. Let an n(k+1)-vector be defined as

$$q = \mathrm{col}\ [q_0^t\ q_1^t\ \ldots\ q_k^t] ,$$

and let $\bar{A}$ be a square matrix of order n(k+1). The matrix $\bar{A}$ is to be partitioned into square submatrices of order n. The submatrices on the main diagonal are set equal to A. The submatrices on the super diagonal are identity matrices. All other elements of $\bar{A}$ are zero. In terms q and $\bar{A}$, (3.25) becomes simply

$$\dot{q} = \bar{A} q \ . \tag{3.26}$$

According to the hypothesis of Theorem 3.3, the initial condition for (3.26) is $q(t_0) = 0$. Since (3.26) is linear, it has the unique solution $q(t) = 0$ for all t in the neighborhood of t_0 for which the matrix M(t) exists.

The matrix M(t) is admissible because M_0 is non-negative definite and symmetric, and $\dot{M}$ may be expressed as an outer product $\beta\beta^t$, where β, from (3.23), is real-valued.

Therefore, the matrix M(t), obtained as a solution of the Riccati equation, satisfies the basic equation (3.8). The coefficient β, evaluated from (3.23) provides a solution to the factorization problem.

Thus far, the existence of an initial condition matrix M_0 satisfying (3.22) has only been postulated. That its existence is not obvious is apparent from the following consideration. In order for (3.22) to be valid, it is necessary that the matrices $\Gamma_k(t_0)$ and $\Phi_k(t_0)$ are consistent in the sense that rank $(\Gamma_k(t_0)) \leq$ rank $(\Phi_k(t_0))$ at some point t_0 in T. For, if this rank condition is violated there can be no matrix M_0 satisfying (3.22). It is demonstrated below that the above rank condition is valid at points dense in T, and furthermore that (3.22) has a (non-unique) solution $M(t_0)$ which is symmetric and non-negative definite.

The following Lemmas will be used to demonstrate the existence of a covariance matrix $M(t_0)$.

Lemma 3.4. Let $r(t,\tau)$ satisfy conditions A1 - A5. Then the non-negative definite matrix $R_k(t)$, defined in equation (3.5), is non-singular at points dense in T.

Proof. The Lemma will be proved by contradiction. Assume that $R_k(t)$ is singular everywhere on a subinterval T' of T. Since the elements of $R_k(t)$ are continuous on T by hypothesis, there exists a continuous vector a(t):

$$a(t) = \text{col}\ [a_o(t)\ a_1(t) \ldots \quad a_k(t)]$$

where $a(t)$ is non-zero everywhere on T', such that

$$a^t R_k a \equiv 0, \text{ on } T' . \tag{3.27}$$

Using (3.7), this identity may be written

$$E[(a^t Y_k)^2] \equiv 0 \text{ on } T',$$

where Y_k was defined by (3.6). Therefore

$$a^t Y_k \equiv 0 \text{ on } T'$$

with probability one. Assume for the present that the component $a_k(t)$ is non zero on T'. Hence, by normalizing the vector $a(t)$ so that $a_k(t) \equiv 1$,

$$y^{(k)}(t) = \sum_{i=o}^{k-1} a_i(t)\, y^{(i)}(t)$$

everywhere on T'. Differentiating this linear constraint yields an expression of the form

$$y^{(k+1)}(t) = \sum_{i=o}^{k} b_i(t)\, y^{(i)}(t) \quad , \tag{3.28}$$

where the functions b_i denote linear combinations of the functions a_i and the derivatives $\dot{a}_i$. In order to verify that the vector $a(t)$ is differentiable, note that from equation (3.5) and Corollaries 3.1 and 3.2, R_k may be expressed as

$$R_k = \Phi_k^t \Gamma_k \ . \tag{3.29}$$

Since Φ_k and Γ_k are differentiable by condition A4, it follows from

(3.27) that $a(t)$ is differentiable.

Equation (3.28) implies that $y^{(k+1)}(t)$ exists on T'. Therefore, from Corollary 3.1, $d_k(t) = 0$ on T', which violates assumption A5. Hence, the matrix $R_k(t)$ is not singular on any subinterval of T.

We now dispense with the assumption that the leading coefficient $a_k(t)$ be non-zero on T'. However, because $a_k(t)$ is continuous on T', then either $a_k(t)$ must be non-zero on some subinterval T" of T', or else $a_k(t) \equiv 0$ on T'. If the former is the case, then the previous argument may still be applied for t restricted to the subinterval T". Suppose the contrary, that $a_k(t) \equiv 0$ on T'. Let j be the largest integer such that $a_j(t)$ is not zero anywhere on I, a subinterval of T'. Certainly, such an $a_j(t)$ exists since the vector $a(t)$ is continuous and never vanishes anywhere on T'. By normalizing $a(t)$ so that $a_j(t)$ is unity on I, we have

$$y^{(j)}(t) = \sum_{i=o}^{j-1} a_i(t)\, y^{(i)}(t) \quad ,$$

for all t in I with probability one. By differentiating this expression $(k+1-j)$ times, an expression for $y^{(k+1)}(t)$ is obtained, which is similar to that in (3.28), and is valid on I. The previous argument may then be applied to establish the Lemma.

An important consequence of Lemma 3.4 is that from (3.29), rank (Φ_k) = rank $(\Gamma_k) = k+1$, at points dense in T.

Lemma 3.5. Let $r(t,\tau)$ satisfy conditions A1 - A4 and let the integer k, appearing in A4, be unspecified. If $d_i(t) = 0$ on T, for $i = 0,1, \ldots n-1$, then $d_i(t) \equiv 0$ on T, for all $i \geq 0$.

Proof. If $d_i(t) = 0$ on T, for $0 \leq i \leq n-1$, then Corollary 3.1 implies that $y^{(i)}(t)$ exists, where $0 \leq i \leq n$. Corollary 3.2 implies that $R_n(t)$ exists and that

$$R_n(t) = \phi_n^t(t)\, \Gamma_n(t) \quad .$$

But, the matrices Φ_n and Γ_n each have n rows. Therefore rank $(R_n) \leq n$, and since the order of R_n is $(n+1)$, $R_n(t)$ is singular everywhere on T. Hence, from the proof of Lemma 3.4, $d_n(t) = 0$ on T. The assertion then follows by induction.

The significance of Lemma 3.5 is that it places an upper bound on the number of computations one is required to perform in the sense that one need never differentiate a function more than n times nor consider a vector of dimension greater than n.

We now proceed to demonstrate the existence of a covariance matrix M_0 satisfying (3.22).

Theorem 3.4. Let $r(t,\tau)$ satisfy conditions A1 - A5. Then there exists a covariance matrix M_0 satisfying (3.22) at a point t_0, where t_0 may be arbitrarily close to a point t, selected arbitrarily in T.

Proof. If t is an arbitrary point in T, then Lemma 3.4 implies that there is a point, t_o, arbitrarily close to t, such that the matrix $R_k(t_o)$ is positive definite. Then, $R_k(t_o)$ has an inverse $R_k^{-1}(t_o)$ which is positive definite. Let a matrix M_o be defined as

$$M_o = \Gamma_k(t_o) R_k^{-1}(t_o) \Gamma_k^{\,t}(t_o) \qquad (3.30)$$

Then clearly, M_o is symmetric and non-negative definite. (In fact, from Lemma 3.5, rank $(M_o) = k+1$.) Since $R_k(t_o)$ is symmetric, (3.29) implies

$$R_k = \Phi_k^t \Gamma_k = \Gamma_k^t \Phi_k \quad .$$

Therefore,

$$M_o \Phi_k(t_o) = \Gamma_k(t_o) R_k^{-1}(t_o) \Gamma_k^t(t_o) \Phi_k(t_o) = \Gamma_k(t_o) R_k^{-1}(t_o) R_k(t_o)$$

$$= \Gamma_k(t_o) \quad .$$

Hence, if M_o is chosen as in (3.30), then M_o satisfies (3.22), and M_o is a covariance matrix.

The matrix M_o given by (3.30) is the unique matrix of minimum rank which satisfies (3.22). For, suppose that some covariance matrix, $M(t_o)$, satisfying (3.22), has rank less than $k+1$. Since

$$\Gamma_k(t_o) = M(t_o) \Phi(t_o) \quad ,$$

then rank $(\Gamma_k(t_o)) < k+1$, which contradicts Lemma 3.4. Therefore $M(t_o)$ cannot have rank less than $k+1$. Suppose $M(t_o)$ is an arbitrary covariance matrix of rank $(k+1)$ satisfying (3.22). Then $M(t_o)$ may be written in the general form

$$M(t_o) = F R_k^{-1}(t_o) F^t \quad , \tag{3.31}$$

where the matrix F has n rows and (k+ 1) columns and has rank (k+ 1). Pre-, and post-multiplying equation (3.31) by Φ_k^t and Φ_k respectively, yields

$$\Gamma_k = F R_k^{-1} (F^t \Phi_k) \quad , \tag{3.32}$$

and

$$R_k = (\Phi_k^t F) R_k^{-1} (F^t \Phi_k) \quad . \tag{3.33}$$

Equation (3.33) implies that

$$R_k = \Phi_k^t F = F^t \Phi_k \quad ,$$

and therefore, from (3.32), $F = \Gamma_k$. Hence M_o given by (3.30) is the unique covariance matrix of rank (k+ 1) satisfying equation (3.22).

One should not conclude from the above discussion that there is only one covariance matrix satisfying equation (3.22). If the rank of $M(t_o)$ is unspecified, then there are an infinite number of covariance matrices $M(t_o)$ satisfying (3.22). The following Corollary will exhibit the general form of a matrix $M(t_o)$.

Corollary 3.4. Let $r(t,\tau)$ satisfy conditions A1 - A5. Let M_o be defined by (3.30), and let N_o be an arbitrary covariance matrix satisfying

$$N_o \Phi_k(t_o) = 0 \quad .$$

Then the matrix $M(t_o) = M_o + N_o$ is a covariance matrix satisfying

(3.22). Furthermore, any covariance matrix satisfying (3.22) may be expressed as $M_o + N_o$. The matrix $M(t_o)$ satisfies
$k+1 \leq \text{rank}\,(M(t_o)) \leq n$.

Proof. Let V be an arbitrary matrix consisting of n rows and (n-k-1) columns. Then, a set of non-unique elements of the matrix V may be determined as a solution of the equation

$$V^t \Phi_k(t_o) = 0 \quad .$$

Define N_o as

$$N_o = V V^t \quad .$$

N_o is recognized as the Gramian matrix of vectors which are rows of V. Therefore N_o is a covariance matrix satisfying $N_o \Phi_k(t_o) = 0$, rank (N_o) = rank (V), and $0 \leq \text{rank } V \leq n\text{-}k\text{-}1$. Therefore, if $M(t_o) = M_o + N_o$, then

$$k+1 \leq \text{rank}\,(M(t_o)) \leq n \quad .$$

Since

$$\begin{aligned} M(t_o)\,\Phi_k(t_o) &= M_o\,\Phi_k(t_o) + V V^t \Phi_k(t_o) \\ &= M_o\,\Phi_k(t_o) = \Gamma_k(t_o) \;, \end{aligned}$$

equation (3.22) is satisfied.

Conversely, let $M(t_o)$ be any covariance matrix satisfying (3.22). Let M_o be defined as before, and now define N_o as

$$N_o = M(t_o) - M_o$$

Then clearly,

$$N_o \Phi_k(t_o) = 0 \quad .$$

Let w be an arbitrary n-vector. Since

$$\Gamma_k^t(t_o)\,\Phi_k(t_o) = R_k(t_o)$$

is a non-singular matrix, then $\Gamma_k^t(t_o)\,w = 0$ if and only if w is orthogonal to the columns of $\Phi_k(t_o)$. Since $M(t_o)$ is a covariance matrix, then

$$0 \le w^t\, M(t_o)\, w^t = w^t(M_o + N_o)w = w^t\, N_o\, w \quad ,$$

if w is orthogonal to the columns of $\Phi_k(t_o)$. If $w = w_1 + w_2$, where w_1 is orthogonal to and w_2 is contained in the subspace generated by the columns of $\Phi_k(t_o)$, then $N_o w_2 = 0$ so that

$$w^t\, N_o\, w = w_1{}^t\, N_o\, w_1 \ge 0$$

for any vector w. Therefore N_o is a covariance matrix.

This section will be concluded by considering several examples.

Example 3.4. Let $d_i^2(t) = 0$ on T for $0 \le i \le n-2$ and let $d_{n-1}^2(t) > 0$ on T. Then on a subinterval T' of T, the square matrix $\Phi_{n-1}(t)$ is non-singular. An initial matrix may now be uniquely determined as

$$\begin{aligned} M(t_o) &= \Gamma_{n-1}(t_o)\, R_{n-1}^{-1}(t_o)\, \Gamma_{n-1}^t(t_o) \\ &= \Gamma_{n-1}(t_o)\, \Phi_{n-1}^{-1}(t_o) \quad , \end{aligned} \tag{3.34}$$

which may be used as an initial condition of the Riccati equation. However, on T', (3.34) yields the unique algebraic expression for M(t). Therefore

$$M(t) = \Gamma_{n-1}(t)\, \Phi_{n-1}^{-1}(t)$$

must coincide with the solution of the Riccati equation on T'. The matrix Φ_{n-1} is recognized as the Wronskian matrix of the basis functions $\phi_i(t)$ of the shaping filter. Since Φ_{n-1} is non-singular on T', the shaping filter may be realized by a single n-th order differential equation defined on T', which has the operational expression

$$L(t,p)\,y = N(t,p)\,u \quad ,$$

where L(t,p) represents an n-th order polynomial in p, the derivative operator, with time-varying coefficients. The operator N(t,p) generally represents a similar polynomial of order less than n. In the present case, Lemma 3.2 implies that the impulse response of the shaping filter, and its first n-2 derivatives are continuous at $\tau = t$. Therefore, [3.10], the operator N(t,p) represents a polynomial of degree zero; i.e., N(t,p) consists only of a time-variable gain.

The present example and the first-order case, considered in Section 2.3, represent two general classes of autocorrelation functions which admit factorization by algebraic means.

Example 3.5. Let

$$r(t,\tau) = \tau/2 - \tau^2/6t, \qquad \text{for } t > \tau \quad ,$$

where $t > 0$, and $\tau > 0$. Set

$$\phi(t) = \begin{bmatrix} 1 \\ -1/t \end{bmatrix} \quad ,$$

and

$$\gamma(t) = \begin{bmatrix} t/2 \\ t^2/6 \end{bmatrix}$$

The functions d_0^2 and d_1^2 may be computed from (3.4) as

$$d_0^2(t) = 0 \qquad \text{for } t > 0 \quad ,$$

and

$$d_1^2(t) = 1/t^2 \quad \text{for } t > 0 \quad .$$

Therefore, this example is recognized as a special case of those considered in Example 3.4. The matrices Γ_1 and Φ_1 may be determined as

$$\Gamma_1(t) = \begin{bmatrix} t/2 & 1/2 \\ t^2/6 & t/3 \end{bmatrix}$$

and

$$\Phi_1(t) = \begin{bmatrix} 1 & 0 \\ -1/t & 1/t^2 \end{bmatrix}$$

The matrix $\Phi_1(t)$ is non-singular for all $t > 0$ and

$$\Phi_1^{-1}(t) = \begin{bmatrix} 1 & 0 \\ t & t^2 \end{bmatrix}$$

The unique expression for the covariance matrix $M(t)$ is given by

$$M(t) = \Gamma_1(t)\, \Phi_1^{-1}(t) = \begin{bmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{bmatrix}$$

The coefficient $\beta(t)$ may be determined either from (3.23), or directly from the diagonal elements of $M(t)$ as

$$\beta(t) = \pm \begin{bmatrix} 1 \\ t \end{bmatrix}$$

The following set of equations describe the shaping filter:

$$\dot{x}(t) = \pm \begin{bmatrix} 1 \\ t \end{bmatrix} u(t)$$

$$y(t) = [\,1 \quad -1/t\,]\, x(t) \quad .$$

An initial value of the state-variable $x(t_o)$ is chosen to be a random variable uncorrelated with the input, such that

$$E[x(t_o)\, x^t(t_o)] = M(t_o) = \begin{bmatrix} t_o & t_o^2/2 \\ t_o^2/2 & t_o^3/3 \end{bmatrix}$$

<u>Example 3.6.</u> Let

$$r(t,\tau) = 3/2\tau + 5\,\tau^2/6t, \quad \text{for } t > \tau \quad ,$$

where $t > 0$, and $\tau > 0$. Set

$$\phi(t) = \begin{bmatrix} 1 \\ 1/t \end{bmatrix}$$

and

$$\gamma(t) = \begin{bmatrix} 3t/2 \\ 5t^2/6 \end{bmatrix}$$

The function $d_o^2(t)$ may be computed as $d_o^2(t) = 4$. Choosing the initial time $t_o = 1$, the matrix

$$M(1) = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$$

satisfies the equation

$$\gamma(1) = M(1)\ \phi(1) \quad ,$$

is positive definite, and may therefore be used as an initial condition for the following Riccati equation:

$$\dot{M}(t) = \frac{1}{4} \left\{ \begin{bmatrix} 3/2 \\ 5t/3 \end{bmatrix} - M(t) \begin{bmatrix} 0 \\ -1/t^2 \end{bmatrix} \right\} \left\{ \begin{bmatrix} 3/2 \\ 5t/3 \end{bmatrix} - M(t) \begin{bmatrix} 0 \\ -1/t^2 \end{bmatrix} \right\}^t$$

One may verify by direct substitution that the matrix

$$M(t) = \begin{bmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{bmatrix}$$

satisfies the Riccati equation and the initial conditions.

3.3 Reduction of the Riccati Equation

In the previous section, a matrix Riccati equation of order n was formulated, the solution of which is M(t), the state-variable covariance matrix of the shaping filter. If one desires to compute the solution of the Riccati equation by numerical or other means, then clearly, computational efficiency will be increased if the order of the differential equation can be reduced. We will show in this section that if the integer k is defined by conditions A4 and A5, then a matrix Riccati equation may be formulated for a square submatrix $M_s(t)$ of M(t), where $M_s(t)$ has order n-k-1, and the remaining elements of M(t) may be obtained from those of $M_s(t)$ by a set of linear relationships which requires the inversion of a matrix of order k+1. The main result is stated in the following Theorem.

Theorem 3.5. Let $r(t,\tau)$ satisfy conditions A1 - A5. Then a matrix Riccati equation may be formulated for $M_s(t)$, a square submatrix of M(t) of order n-k-1, which is valid on a subinterval of T.

Proof. Since the matrix $\Phi_k(t)$ has rank k+1 at points dense in T, then there is a subinterval T', of T on which $\Phi_k(t)$ has rank k+1 everywhere. Since $\Phi_k(t)$ is continuous, there is a subinterval, T", of T' on which a square submatrix $\Phi_{k1}(t)$ has rank k+1 everywhere. By re-ordering the rows of $\Phi_k(t)$, a convenient partition of $\Phi_k(t)$ may be established as

$$\Phi_k(t) = \begin{bmatrix} \Phi_{k1}(t) \\ --- \\ \Phi_{k2}(t) \end{bmatrix}$$

where $\Phi_{k1}(t)$ has rank $k+1$ everywhere on a subinterval of T. The submatrix Φ_{k2} has $n-k-1$ rows. The matrix $\Gamma_k(t)$ may also be partitioned into submatrices Γ_{k1} and Γ_{k2}, having $k+1$ and $n-k-1$ rows respectively, and a similar partition holds for the vectors $\phi^{(k+1)}$, $\gamma^{(k+1)}$, and β.

Theorem 3.3 established the validity of the equation

$$\Gamma_k(t) = M(t)\, \Phi_k(t) \quad , \tag{3.35}$$

to be satisfied by an admissible matrix M(t). Introducing conformable submatrices $M_1(t)$, $M_2(t)$, and $M_s(t)$, of M(t), yields for (3.25)

$$\begin{aligned} \Gamma_{k1} &= M_1 \Phi_{k1} + M_2 \Phi_{k2} \\ \Gamma_{k2} &= M_2^t \Phi_{k1} + M_s \Phi_{k2} \ . \end{aligned} \tag{3.36}$$

Inverting the previous equations yields

$$\begin{aligned} M_1 &= \Gamma_{k1} \Phi_{k1}^{-1} - M_2 \Phi_{k2} \Phi_{k1}^{-1} \\ M_2^t &= \Gamma_{k2} \Phi_{k1}^{-1} - M_s \Phi_{k2} \Phi_{k1}^{-1} \ , \end{aligned} \tag{3.37}$$

which indicates that once M_s is determined, M_1 and M_2 may be found in terms of M_s.

In terms of the above submatrices, (3.23) may be rewritten as

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \frac{\begin{bmatrix} \gamma_1^{(k+1)} \\ \gamma_2^{(k+1)} \end{bmatrix} - \begin{bmatrix} M_1 & M_2 \\ M_2^t & M_s \end{bmatrix} \begin{bmatrix} \phi_1^{(k+1)} \\ \phi_2^{(k+1)} \end{bmatrix}}{d_k} \ . \tag{3.38}$$

By performing the indicated matrix multiplication and substituting the expression for M_2^t derived in (3.37) into the above equation, we obtain the following equation for β_2:

$$\beta_2 = \frac{g - M_s f}{d_k} , \tag{3.39}$$

where

$$g = \gamma_2^{(k+1)} - \Gamma_{k2} \Phi_{k1}^{-1} \phi_1^{(k+1)} ,$$

and

$$f = \phi_2^{(k+1)} - \Phi_{k2} \Phi_{k1}^{-1} \phi_1^{(k+1)} .$$

Noting that $\dot{M}_s = \beta_2 \beta_2^t$, (3.39) may be transformed into the following matrix Riccati equation, similar in form to that in (3.21), but of degree n-k-1 :

$$\dot{M}_s = \frac{(g - M_s f)(g - M_s f)^t}{d_k^2} , \tag{3.40}$$

which is the desired differential equation valid on the subinterval T" of T.

The proof of Theorem 3.5 depends upon the definition of the interval T" on which a submatrix Φ_{k1} is non-singular. It is possible to extend the domain of definition of the matrix Φ_{k1} by noting that time-varying elementary row operations [3.10] may be performed on $\Phi_k(t)$ in such a way that the first $k+1$ rows of the row-permuted matrix are linearly independent vectors for each t in T'. Denote by A(t) the (non-singular) matrix of elementary row operations. Then by defining

$$\overline{\Phi}_k = A\,\Phi_k$$

$$\overline{\Gamma}_k = A\,\Gamma_k$$

and

$$\overline{M} = A\,M\,A^t \;,$$

the equation

$$\overline{\Gamma}_k = \overline{M}\,\overline{\Phi}_k$$

is valid on T' and may be partitioned as before. If the derivation used in the proof of Theorem 3.5 is repeated, a Riccati equation may be derived for the submatrix $\overline{M}_s$ which is valid everywhere on T', the interval on which $\Phi_k(t)$ has rank $k+1$.

In certain special cases, the matrix $\Phi_{k1}(t)$ will be defined and non-singular on the entire interval of interest, T. A large and important class of such special cases concerns shaping filters which may be represented as a single n-th order linear differential equation. It is well known, [3.12], that the necessary and sufficient conditions that the state-variable equations (2.5) admit an equivalent representation as a single n-th order differential equation on the interval T is that the Wronskian matrix of the vector $\phi(t)$, i.e., the matrix $\Phi_{n-1}(t)$, exists and is non-singular on T. If such is the case, then the matrix $\Phi_k(t)$, which consists of the first $k+1$ columns of $\Phi_{n-1}(t)$, must have rank $k+1$ everywhere on T. Therefore, the interval T' coincides with T, and the above proof and discussion may be applied in order to show that (3.40) is valid everywhere on T.

Omitted from Theorem 3.5 is the fact that an admissible solution $M_S(t)$ of (3.40), together with the transformations in (3.37), determine an admissible matrix M(t), which is a solution of (3.21). The proof of this statement is elementary, requiring differentiation of the partitioned matrix M(t) and substituting equations (3.37) and (3.40). The algebraic details are tedious and are therefore omitted. Note that the question of the existence of an initial condition matrix for (3.40) is of no concern, since $M_S(t_o)$ may be chosen as the appropriate submatrix of $M(t_o)$ which, from Theorem 3.4, is known to exist.

As an example of the reduction technique, Example 3.6 shall be reconsidered.

Example 3.6 (continued). Let

$$r(t,\tau) = 3/2\tau + 5\,\tau^2/6t, \quad \text{for } t > \tau \ .$$

The function $d_o^2(t)$ has been computed as $d_o^2(t) = 4$; therefore $k = 0$. The required submatrices, defined in this section, become

$$\Phi_{k1} = 1 \ ,$$

$$\Phi_{k2} = 1/t \ ,$$

$$\Gamma_{k1} = 3t/2 \ ,$$

$$\Gamma_{k2} = 5t^2/6 \ ,$$

$$\gamma_2^{(k+1)} = 5t/3 \ ,$$

and

$$\phi_1^{(k+1)} = 0 \ .$$

The functions f(t) and g(t) become:

$$f(t) = -1/t^2 \quad ,$$

and

$$g(t) = 5t/3 \quad .$$

The reduced matrix Riccati equation (3.40) becomes

$$\dot{M}_S = (-1/t^2 + M_S \, 5t/3)^2/4 \quad ,$$

where $M_S(t)$ is a scalar. To be consistent with the previous consideration of this example, let $t_o = 1$ and $M_S(t_o) = 1/3$. One may verify by direct substitution that the function,

$$M_S(t) = t^3/3 \quad ,$$

satisfies the Riccati equation and the initial condition. Substitution of the solution $M_S(t) = t^3/3$ into (3.37) yields the following for the submatrices M_1 and M_2:

$$M_1(t) = t \quad ,$$

and

$$M_2(t) = t^2/2 \quad ,$$

which is identical with the solution obtained in the previous section.

In addition to the reduction of order demonstrated above, it may be shown that the matrix $M_S(t)$ is related to the solution of a set of 2(n-k-1) linear differential equations which are associated with the Riccati equation. Let $z_a(t)$ and $z_b(t)$ denote vectors of dimension n-k-1 which are a solution of the following linear equation:

$$\begin{bmatrix} \dot{z}_a \\ \dot{z}_b \end{bmatrix} = \frac{1}{d_k^2} \begin{bmatrix} g \\ f \end{bmatrix} [-f^t \quad g^t] \begin{bmatrix} z_a \\ z_b \end{bmatrix} \tag{3.41}$$

Let $Z(t)$ represent a fundamental matrix solution of (3.41), such that $Z(t_o) = I$, and partition Z into square submatrices of order $n-k-1$, so that

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$$

It is well known [3.13] that $M_s(t)$ may be determined from the submatrices of $Z(t)$ and from an initial matrix $M_s(t_o)$ in the following way:

$$M_s(t) = [Z_1(t)\, M_s(t_o) + Z_2(t)][Z_3(t)\, M_s(t_o) + Z_4(t)]^{-1} \tag{3.42}$$

If one does not desire to reduce the order of the Riccati equation, then the vectors f and g may be replaced by $\phi^{(k+1)}$ and $\gamma^{(k+1)}$ respectively in (3.41) yielding a set of linear differential equations of order 2n. Equation (3.42) remains valid upon replacing $M_s(t)$ and $M_s(t_o)$ by $M(t)$ and $M(t_o)$ respectively.

It has been shown by Levin, [3.13], that $M_s(t)$ as obtained from (3.42) is uniquely determined by specifying $M_s(t_o)$, and $M_s(t)$ is identical to the solution of the Riccati equation (3.40). Note that the matrix inversion indicated in (3.42) exists for $t=t_o$, and by continuity, in a neighborhood of t_o, which reflects the fact that the existence theorem for ordinary differential equations [3.9] guarantees only a local solution.

It is interesting at this point to compare the results of this section with the results obtained by Darlington [3.14]. Darlington considered shaping filters which are realizable by a single n-th order linear differential equation, and showed for certain cases that the solution of the factorization problem corresponds to the solution of a single linear differential equation of order 2(n-k-1). For Darlington, the integer k is interpreted to mean $\frac{\partial^i}{\partial t^i} h(t,\tau)$ is continuous at $\tau = t$ for $0 \leq i \leq k$ and $\frac{\partial^k}{\partial t^k} h(t,\tau)$ has a simple discontinuity at $\tau = t$, where $h(t,\tau)$ is the impulse response of the shaping filter. The results of Section 3.1 imply that k, as (implicitly) defined by Darlington, is identical with k as defined by conditions A4 and A5 in the present work.

It was shown above by utilizing a linear constraint imposed on the covariance matrix M(t), that the coefficients of a shaping filter represented by (2.5) can be determined via the solution of a set of linear equations of order 2(n-k-1), thereby illustrating a point of coincidence of the present work and the work of Darlington.

Although the results of this section were introduced by an appeal for computational efficiency, it is clear that, depending on the value of k, whatever efficiency is gained by reducing the order of the Riccati equation may be lost by requiring inversion of a matrix function of order k+1. The same comments apply if one were to compute solutions to equations (3.41) and (3.42). In this case, the differential equation has order 2(n-k-1), and the matrix function to be inverted has order n-k-1.

Clearly, if any computational scheme is to be effective, one must be assured that the desired solution will exist in more than an unspecified neighborhood of t_0. The next chapter will be concerned with the determination of global solutions of the factorization problem.

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CHAPTER IV

REALIZATION OF THE SHAPING FILTER:

GLOBAL EXISTENCE

4.0 Introduction

The results of Chapter III are limited in the sense that a solution of a Riccati equation is guaranteed to exist only in a local neighborhood of the initial time. In this Chapter, sufficient conditions will be formulated which allow the Riccati equation, and hence, the factorization problem, to have a global solution.

An interesting and important result presented here is the formulation of a set of criteria, not including A3 the non-negative definite condition, which are necessary and sufficient to describe the class of autocorrelation functions under consideration.

If the factorization problem has a global solution, then stability of the shaping filter is an important consideration. Stability will be defined in an appropriate way and a sufficient condition for the stability of the shaping filter will be established.

4.1. Global Solution of the Factorization Problem

The standard existence theorem for ordinary differential equations [4.1] implies that a solution of a non-linear differential equation exists only in a local neighborhood of the initial time. It is well-known that a solution of a non-linear differential equation may exhibit the phenomenon of finite escape time; i.e., the solution may become unbounded at a finite time after the initial time. Finite escape is relevant to the factorization problem because the Riccati equation is non-linear.

The following example considers an apparently well-behaved autocorrelation function for which the corresponding covariance matrix, $M(t)$, exhibits the finite escape time phenomenon.

Example 4.1. Let

$$r(t,\tau) = \begin{cases} -\max(t,\tau) , & \text{for } t \text{ and } \tau < 0 , \\ 0 , & \text{for } t \text{ or } \tau > 0 . \end{cases}$$

The functions ϕ and γ may be determined as

$$\phi(t) = \begin{cases} t , & \text{for } t < 0 \\ 0 , & \text{for } t \geq 0 , \end{cases}$$

and

$$\gamma(t) = 1 , \quad \text{for all } t .$$

Although the results of Chapter III may be used to formulate a Riccati equation for $M(t)$, the present example corresponds to the first-order case, which was considered in Section 2.3, and $M(t)$ may be obtained algebraically as

$$M(t) = -1/t \quad , \text{ for } t < 0 \quad .$$

Clearly, M(t) escapes at $t = 0$. The coefficient of the shaping filter may be determined as

$$\beta(t) = [\, M(t)\,]^{\frac{1}{2}} = \pm\, 1/t \quad ,$$

which also escapes at $t = 0$.

Clearly, such unbounded behavior of M(t) is undesirable, especially if simulation is contemplated. The present objective is to establish conditions which suffice to prevent finite escape of M(t). Such a condition is readily available in the first-order case, as was shown in Section 2.3. Indeed, if the function $\phi(t)$ is such that $\phi(t)$ never vanishes indefinitely, then the results of Section 2.3 indicate that M(t) may be defined for all t, where $t_o \leq t < \infty$, and $\phi(t_o) \neq 0$.

In order to effect a generalization to the multi-dimensional case, it is both convenient and natural to utilize the concept of complete observability. Kalman [4.2] has shown that the shaping filter, represented by equation (2.5), is completely observable if and only if for any t_1, there is a finite $t_2 > t_1$ such that the functions $\phi_i(t)$, for $i = 1,2, \ldots n$, are linearly independent on the interval $[t_1, t_2]$. Therefore, one may determine directly from the given data whether the shaping filter will be completely observable. In the first-order case, complete observability reduces to the non-vanishing property of $\phi(t)$, discussed above.

The following Theorem[†] shows that complete observability is sufficient to insure that the desired covariance matrix M(t) is finite provided that M_0 is sufficiently small.

Theorem 4.1. Let $T = [t_0, \infty)$, and let $r(t,\tau)$ satisfy conditions A1 and A2, let $r(t,t)$ and $\phi(t)$ be finite for $t_0 \le t \le \infty$, and let $r(t,\tau)$ admit factorization so that

$$r(t,\tau) = \phi^t(t)\, M[\min(t,\tau)]\, \phi(\tau) \quad ,$$

where M(t) is admissible. Then M(t) is finite for all t in T provided that values of $M(t_0)$ are suitably restricted and that the shaping filter is completely observable.

Proof. Let x(t) be a vector-valued random process for which

$$M(t) = M[x(t)\, x^t(t)] \quad ,$$

and define a scalar-valued process $v(\lambda)$ as

$$v(\lambda) = \phi^t(\lambda)\, x(t) \quad ,$$

for fixed t and all $\lambda \ge t$. Then, from Theorem 2.1, the function

$$E[v(\lambda)\, v(\xi)] = \phi^t(\lambda)\, M(t)\, \phi(\xi) \qquad (4.1)$$

is an autocorrelation function. By hypothesis, M(t) is non-decreasing so that

$$E[v^2(\lambda)] = \phi^t(\lambda)\, M(t)\, \phi(\lambda) \le r(\lambda,\lambda) \qquad (4.2)$$

for $\lambda \ge t$. The Schwarz inequality states that

† Theorem 4.1 was originally stated without proof in [4.3].

$$(E[v(\lambda)\, v(\xi)])^2 \le E[v^2(\lambda)] \; E[v^2(\xi)] \quad ,$$

which may be applied to (4.1) and (4.2) to yield

$$|\phi^t(\lambda)\, M(t)\, \phi(\xi)| \le [r(\lambda,\lambda)\, r(\xi,\xi)]^{\frac{1}{2}} < \infty \quad , \tag{4.3}$$

for all $\lambda \ge t$, and $\xi \ge t$.

Pre-multiplying and post-multiplying equation (4.3) by $\phi(\lambda)$ and $\phi^t(\xi)$ respectively, yields

$$|\mathrm{tr}\; \phi(\lambda)\, \phi^t(\lambda)\, M(t)\, \phi(\xi)\, \phi^t(\xi)| \le |\mathrm{tr}\, \phi(\lambda)\, [r(\lambda,\lambda)\, r(\xi,\xi)]^{\frac{1}{2}}\, \phi^t(\xi)| \; , \tag{4.4}$$

where "tr" abbreviates trace.

Equation (4.4) may be integrated with respect to both variables λ and ξ, between the limits t and t'. On the left side of equation (4.4), the double integral may be moved within the magnitude sign and then commuted with the trace operation without invalidating the inequality.

Define the non-negative definite Gramian matrix of the functions $\phi_i(t)$ as

$$G(t,t') = \int_t^{t'} \phi(\lambda)\; \phi^t(\lambda)\, d\lambda \tag{4.5}$$

It is well-known [4.4] that singularity of the Gramian matrix is necessary and sufficient to determine the linear dependence of the functions $\phi_i(t)$ on the interval $[t,t']$. However, by assumption, a finite $t' > t$ may always be found so that $G(t,t')$ is non-singular.

Let the scalar $\rho(t,t')$ denote the result of integrating the right-hand side of equation (4.4), i.e.,

$$\rho(t,t') = \int_t^{t'}\int_t^{t'} \left| \phi^t(\lambda) \, [\, r(\lambda,\lambda)\, r(\xi,\xi)\,]^{\frac{1}{2}} \, \phi(\xi) \right| d\lambda \, d\xi \tag{4.6}$$

Since the integrand of equation (4.6) is finite, equations (4.4), (4.5) and (4.6) yield

$$|\operatorname{tr} G(t,t')\, M(t)\, G(t,t')| \leq \rho(t,t') < \infty \tag{4.7}$$

Since the left side of equation (4.7) is bounded, the magnitude sign may be omitted. One may then show, using elementary properties of non-negative definite symmetric matrices, that

$$\operatorname{tr} M(t) \leq \rho(t,t') \operatorname{tr}^2 G^{-1}(t,t') < \infty$$

which must hold for all t' for which G^{-1} is non-singular. Therefore,

$$\operatorname{tr} M(t) \leq \inf_{t'} \rho(t,t') \operatorname{tr}^2 G^{-1}(t,t') < \infty \tag{4.8}$$

In particular, $\operatorname{tr} M(t_0)$ cannot violate the inequality (4.8), for then $M(t)$ must have a finite escape time, thus violating (4.8).

Theorem 4.1 states that the covariance $M(t)$ is finite provided that $M(t_0)$ is sufficiently small. Clearly, (4.8) provides a necessary but not sufficient upper bound on $\operatorname{tr} M(t_0)$. The following Example explores the finite escape phenomenon for a stationary random process.

Example 4.2. Let

$$r(t,\tau) = \frac{4}{3} e^{-|t-\tau|} - \frac{5}{12} e^{-2|t-\tau|} ,$$

and let

$$\gamma(t) = \begin{bmatrix} \frac{4}{3} e^{t} \\ \frac{-5}{12} e^{2t} \end{bmatrix} \quad \text{and} \quad \phi(t) = \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix}$$

Application of the reduction technique of Section 3.3 yields the following Riccati equation for the scalar covariance $M_S(t)$:

$$\dot{M}_S(t) = \left(\frac{5}{4}e^{2t} - M_S(t)\, e^{-2t}\right)^2 . \tag{4.9}$$

From (3.22) and Corollary 3.4, the minimum value of $M_S(0)$ is $M_S(0) = 75/396$. The general solution of (4.9),

$$M_S(t) = \frac{\frac{25}{4}e^{4t} - \frac{1}{4}\left(\frac{M_S(0) - 25/4}{M_S(0) - 1/4}\right)e^{10t}}{1 - \left(\frac{M_S(0) - 25/4}{M_S(0) - 1/4}\right)e^{6t}} , \tag{4.10}$$

is illustrated in Figure 4-1 for several values of $M_S(0)$. If $M_S(0) > 25/4$, then the denominator of (4.10) will vanish at a finite time $t > 0$, so that $M_S(t)$ will have a finite escape time. If $75/396 \leq M_S(0) \leq 25/4$, then $M_S(t)$ will be well-behaved. However, the upper bound given by (4.8) is $\operatorname{tr} M(0) \leq 3484$. But corresponding to $M_S(0) \leq 25/4$, we have from (3.37) that $\operatorname{tr} M(0) \leq 57/4$, illustrating that (4.8) is necessary but not sufficient to insure a finite $M(t)$.

Two solutions of (4.9) are of special interest. Corresponding to $M_S(0) = 25/4$ is the solution

$$M_{S1}(t) = 25/4\; e^{4t} ,$$

and corresponding to $M_S(0) = 1/4$ is the solution

$$M_{S2}(t) = 1/4\; e^{4t} .$$

Evidently from (4.10)

$$\lim_{t \to \infty} M_S(t) = 1/4\; e^{4t}$$

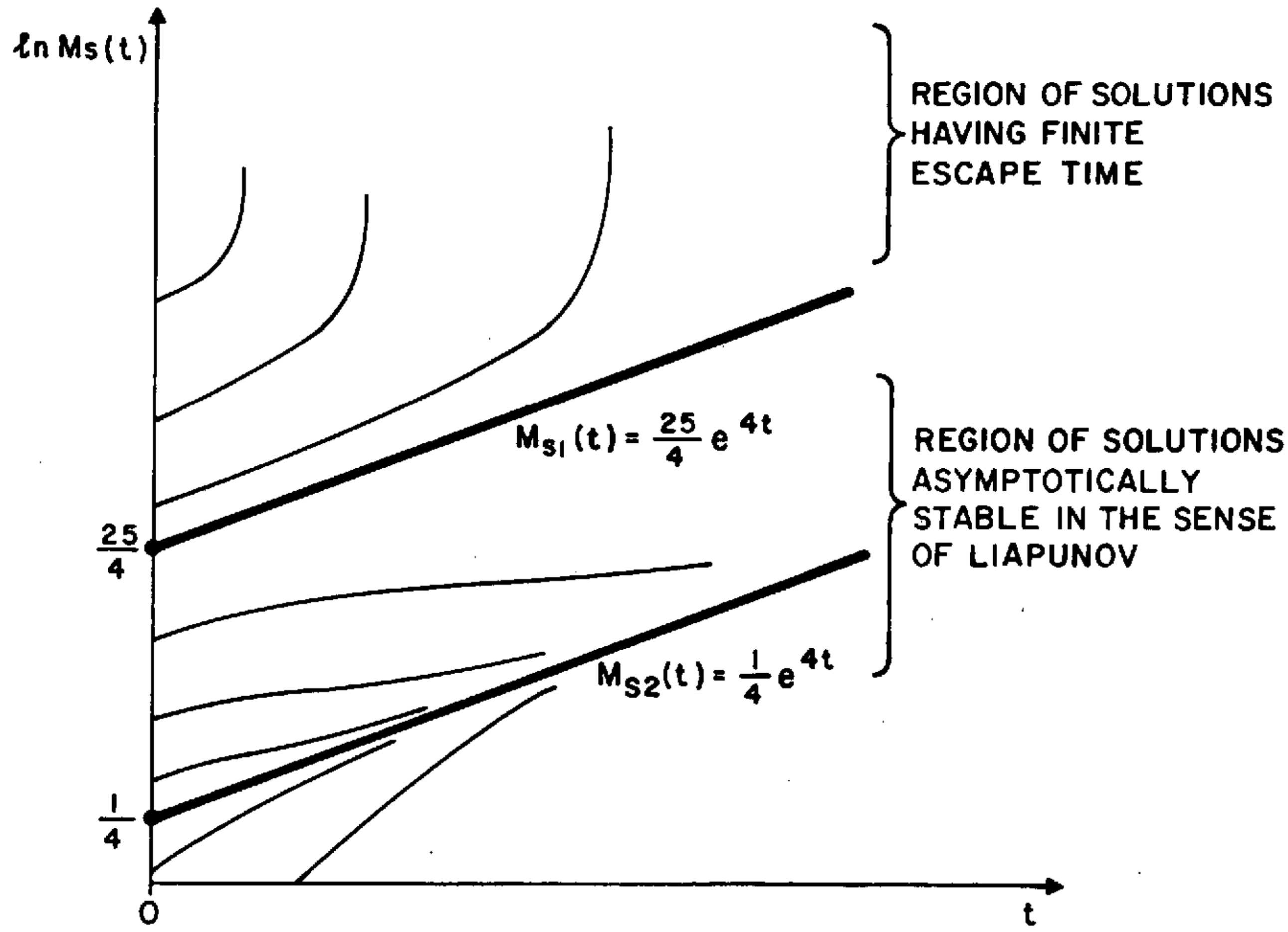

FIG. 4-1 ILLUSTRATING BEHAVIOR OF SOLUTIONS $M_s(t)$ OF EQUATION (4.9) FOR VARIOUS INITIAL VALUES $M_s(0)$. SOLUTIONS $M_{s1}(t)$ AND $M_{s2}(t)$ CORRESPOND TO TIME-INVARIANT SHAPING FILTERS

for all $M_S(0) < 25/4$. Therefore the solution $M_{S1}(t)$ corresponding to $M_S(0) = 25/4$ is unstable in the sense of Liapunov and represents a separatrix, and the solution $M_{S2}(t)$ corresponding to $M_S(0) = 1/4$ is asymptotically stable. The transfer function of the shaping filter corresponding to $M_{S1}(t)$ is

$$H_1(s) = \frac{s-3}{(s+1)(s+2)} \quad ,$$

and the transfer function of the shaping filter corresponding to $M_{S2}(t)$ is

$$H_2(s) = \frac{s+3}{(s+1)(s+2)} \quad .$$

The transfer functions H_1 and H_2 represent the two realizable solutions of the factorization problem which are produced by the Bode-Shannon technique [4.5]. But since $M_{S1}(t)$ is unstable, then $H_1(s)$ cannot be realized using the present factorization technique. Note further that $H_1^{-1}(s)$ is unrealizable and that $H_2^{-1}(s)$ is realizable (with the exception of a required differentiation.) Therefore the solutions of (4.9) for $M_S(0) < 25/4$ determine shaping filters which are asymptotically time-invariant and which converge to a shaping filter having the transfer function $H_2(s)$. The whitening filter corresponding to $H_2(s)$ is stable.

The above example established a relation between the finite escape time phenomenon and some asymptotic properties of the shaping filter. This example suggests that a similar relation may hold for stationary processes in general, and possibly for some non-stationary processes. Future research will examine this question.

Theorem 4.1 is difficult to apply because (4.8) is only a necessary upper bound for $\operatorname{tr} M(t)$. A sufficient bound for $\operatorname{tr} M(t)$ related to the solution of a scalar differential equation, is presented below.

Theorem 4.2. Let $r(t,\tau)$ satisfy A1 - A5 and let $V(t)$ satisfy the differential equation

$$\dot{V} = \frac{\|\phi^{(k+1)}\|^2}{d_k^2} V^2 + \frac{2\,|\phi^{(k+1)t}\gamma^{(k+1)}|}{d_k^2} V + \frac{\|\gamma^{(k+1)}\|^2}{d_k^2}\,, \tag{4.11}$$

where $\|\cdot\|$ represents the Euclidean norm. Let $V_{max}(t_o)$ be the largest value of $V(t_o)$ such that the solution of (4.11) corresponding to $V(t_o)$ has no finite escape time for $t \geq t_o$. Then any solution $M(t)$ of (3.21) is finite for all $t \geq t_o$ if $\operatorname{tr} M(t_o) < V_{max}(t_o)$.

Proof. Using elementary properties of non-negative matrices, we may transform the Riccati equation (3.21) into the following inequality:

$$\frac{d}{dt}\operatorname{tr} M \leq \frac{\|\phi^{(k+1)}\|^2}{d_k^2}\operatorname{tr}^2 M + \frac{2\,|\phi^{(k+1)t}\gamma^{(k+1)}|}{d_k^2}\operatorname{tr} M + \frac{\|\gamma^{(k+1)}\|^2}{d_k^2}. \tag{4.12}$$

Let $V(t)$ be the solution of (4.11) corresponding to the initial condition $V_{max}(t_o)$ and let $\operatorname{tr} M(t_o) < V_{max}(t_o)$. Subtracting (4.12) from (4.11) yields

$$\frac{d}{dt}(V - \operatorname{tr} M) \geq \frac{\|\phi^{(k+1)}\|^2}{d_k^2}(V - \operatorname{tr} M)^2$$
$$+ \frac{2(\|\phi^{(k+1)}\|^2 \operatorname{tr} M + |\phi^{(k+1)t}\gamma^{(k+1)}|)}{d_k^2}(V - \operatorname{tr} M)\quad .$$

Note that $\frac{d}{dt}(V - \mathrm{tr}\, M)$ is non-negative if and only if $V - \mathrm{tr}\, M$ is non-negative, so that $V - \mathrm{tr}\, M$ cannot change sign. Hence, if $V_{max}(t_o) - \mathrm{tr}\, M(t_o) > 0$, then $V(t) - \mathrm{tr}\, M(t) > 0$ for all $t > t_o$.

Theorem 4.2 requires the solution of the Riccati equation (4.11) in order to provide an upper bound for $\mathrm{tr}\, M(t_o)$. However, if the coefficients of (4.11) are bounded by exponential functions, as is the case for some stationary random processes, then an explicit upper bound may be obtained.

Let $\frac{\|\gamma^{(k+1)}(t)\|^2}{d_k^2(t)} \le A_1^2 e^{\alpha t}$, $\frac{\|\phi^{(k+1)}(t)\|^2}{d_k^2(t)} \le A_2^2 e^{-\alpha t}$, and $\frac{\|\phi^{(k+1)}(t)\, \gamma^{(k+1)}(t)\|}{d_k^2(t)} \le A_1 A_2$, for some A_1, A_2 and $\alpha > 0$. Then (4.12) becomes

$$\frac{d}{dt}\, \mathrm{tr}\, M(t) \le (A_1 e^{\alpha t/2} + A_2 e^{-\alpha t/2}\, \mathrm{tr}\, M(t))^2 \tag{4.13}$$

Equation (4.13) may be integrated in closed form, with the result that if

$$\mathrm{tr}\, M(0) \le \frac{\alpha - 2A_1A_2}{2A_2} + \left(\frac{\alpha^2 - 4A_1A_2\alpha}{4A_2^4}\right)^{\frac{1}{2}} , \tag{4.14}$$

where

$$\alpha > 4 A_1 A_2$$

then $M(t)$ will be finite for all $t \ge 0$. But, from Corollary 3.4,

$$\mathrm{tr}\, M(0) \ge \mathrm{tr}\, \Gamma_k(0)\, R_k^{-1}(0)\, \Gamma_k^t(0) . \tag{4.15}$$

Therefore, if (4.14) is to provide a meaningful upper bound, (4.14) and (4.15) must be consistent. Henceforth, we will assume for simplicity that solutions $M(t)$ of (3.21) are finite for all $t > t_o$.

In order to realize the shaping filter, one needs assurance, not only that M(t) is finite, but that the coefficient $\beta(t)$ is finite. As the following first-order example shows, M(t) may be finite but $\beta(t)$ may become infinite at a finite time.

Example 4.3. Let $r(t,\tau) = \phi(t)\gamma(\tau)$ for $t \geq \tau$, where

$$\phi(t) = |t|^{\frac{1}{2}}, \text{ and}$$

$$\gamma(t) = \begin{cases} 2\,|t_o t|^{\frac{1}{2}} - 2\,|t|, & \text{for } t_o \leq t \leq 0 \\ 2\,|t_o t|^{\frac{1}{2}} + 2\,|t|, & \text{for } t_o \leq 0 \leq t \end{cases},$$

and for t_o negative. Using the technique of Section 2.3, we may calculate M(t) as

$$M(t) = \frac{\gamma(t)}{\phi(t)} = \begin{cases} 2\,|t_o|^{\frac{1}{2}} - 2\,|t|^{\frac{1}{2}}, & \text{for } t_o \leq t \leq 0 \\ 2\,|t_o|^{\frac{1}{2}} + 2\,|t|^{\frac{1}{2}}, & \text{for } t_o \leq 0 \leq t \end{cases}.$$

Clearly, M(t) is non-negative, non-decreasing, and finite for $t \geq t_o$. Differentiating the above expression yields

$$\dot{M}(t) = |t|^{-\frac{1}{2}}, \text{ for } t \geq t_o,$$

and

$$\beta(t) = [\dot{M}(t)]^{\frac{1}{2}} = |t|^{-1/4},$$

which is unbounded for $t = 0$.

From Lemma 3.2,

$$d_o(t) = \phi(t)\,\beta(t) = |t|^{1/4},$$

which is zero for $t = 0$.

The following Theorem presents conditions which are sufficient to guarantee that the function $\beta(t)$ is finite.

Theorem 4.3. Let $r(t,\tau)$ satisfy conditions A1 -A5, and let $M(t)$ be defined for all $t \geq t_0$. Then $\beta(t)$ is finite for all $t \geq t_0$.

Proof. From Theorem 3.3, the covariance matrix $M(t)$ may be obtained as a solution to the matrix Riccati equation (3.21), and by assumption, $M(t)$ is finite. The coefficient $\beta(t)$ is determined from equation (3.23) as

$$\beta(t) = \frac{\gamma^{(k+1)}(t) - M(t)\,\phi^{(k+1)}(t)}{d_k(t)} .$$

The functions $\gamma^{(k+1)}(t)$ and $\phi^{(k+1)}(t)$ are finite on T since, by condition A4, they are continuous on T. The function $d_k(t)$ does not vanish on T, by condition A5. Therefore $\beta(t)$ is finite on T.

The condition that $d_k(t)$ be non-vanishing on T is only a sufficient condition as may be verified by reconsidering Example 2.1.

Example 2.1, (continued)

$$r(t,\tau) = \tau \cos \tau \cos t .$$

Letting

$$\phi(t) = \cos t ,$$

and

$$\gamma(t) = t \cos t ,$$

it was previously established that

$$M(t) = t \quad ,$$

and

$$\beta(t) = \pm 1 \quad ,$$

which are finite. However,

$$d_o(t) = \phi(t)\,\beta(t) = \pm \cos t \quad ,$$

which vanishes at $t = (2n-1)\pi/2$, for $n = 1, 2, \ldots$.

We have introduced sufficient conditions in Theorems 4.1 - 4.3 which guarantee that the covariance matrix $M(t)$, and the coefficient $\beta(t)$ must be finite. Therefore, a solution of the Riccati equation, and of the reduced Riccati equation discussed in Section 3.3, may be defined globally, and the factorization problem has a global solution.

Although emphasis in Chapter III has been given to questions of existence and realizability, the results of this section allow the possibility of practical computation of the solution of the Riccati equation. Digital computer programs designed for numerical integration of a matrix Riccati equation are in existence, [4.6], [4.7]. Therefore, one may consider the preceding results not only as an existence theory, but as a constructive factorization technique which will yield numerical values of the coefficients of the shaping filter.

4.2. Characterization of Autocorrelation Functions

With the assumption that $M(t)$ is finite added to the list of basic conditions A1 -A5, it is possible to characterize the class of autocorrelation under consideration in a simple fashion which obviates explicit reference to condition A3, the non-negative definite condition.

Such a characterization is desirable because, with the exception of A3, conditions A1 - A5 provide simple criteria which must be satisfied by the given functions $\phi(t)$ and $\gamma(t)$; tests for these criteria may be devised which require only a finite number of calculations. On the other hand, the definition of the non-negative definite property, given in A3, requires calculation of an infinite number of determinants of all orders. Clearly, a test based directly on the definition of the non-negative definite property will never terminate.

If the given autocorrelation function corresponds to a stationary random process, i.e., if $r(t,\tau)$ is a function of the difference $|t - \tau|$, then the non-negative definite property has a simple physical interpretation. Bochner's theorem [4.8] states that a continuous, symmetric function $r(t - \tau)$ is an autocorrelation function if, and only if $r(t - \tau)$ is the Fourier transform of an everywhere non-negative function. In other words, $r(t - \tau)$ must correspond to a physically meaningful, i.e., non-negative, power spectrum. Clearly, the methods of Fourier analysis and the use of appropriate approximation techniques may provide a finite test to be applied to functions $r(t-\tau)$.

In the non-stationary case, Fourier or frequency-domain analysis is generally inapplicable. However, there is a result based on spectral decomposition, which bears some resemblance to Bochner's theorem. Consider the following integral equation:

$$\int_T r(t,\tau)\,\theta_i(\tau)\,d\tau = \lambda_i\,\theta_i(t) \tag{4.16}$$

Solutions $\theta_i(t)$ are the eigenfunctions of the kernel $r(t,\tau)$, and the scalars λ_i are the eigenvalues. It is well-known, [4.9], that the kernel $r(t,\tau)$ is non-negative definite if, and only if all the eigenvalues λ_i are non-negative. But, there are, in general, an infinite number of functions $\theta_i(t)$ and scalars λ_i satisfying (4.16). In addition, solutions $\theta_i(t)$ are generally unavailable in closed form. Therefore, a test based on solving (4.16) will not terminate. Furthermore, such a test has an additional shortcoming. Suppose that $r(t,\tau)$ is given as the result of a process of approximation and interpolation performed on a set of data points. Then it is possible that $r(t,\tau)$ may satisfy A3 on $T' \times T'$, where T' is a subinterval of T, but may not satisfy A3 on $T \times T$. In such a case, some of the eigenvalues λ_i will be negative, indicating that $r(t,\tau)$ does not satisfy A3 on $T \times T$. Inorder to determine the sub-square $T' \times T'$ on which $r(t,\tau)$ is non-negative definite, it is necessary to investigate (4.16) on every sub-interval of T, which process does not terminate.

In the present case, the class of functions $r(t,\tau)$ is not completely general. Physically reasonable limitations have been imposed on $r(t,\tau)$ to the effect that $r(t,\tau)$ is separable, and differentiable a finite number of times. As will be shown below, these restrictions on the class of functions $r(t,\tau)$ allow the formulation of easily applied criteria, which are necessary and sufficient to determine whether a particular function $r(t,\tau)$ is non-negative definite.

<u>Theorem 4.4.</u> Let $T = (t_o, t_1)$, and let $r(t,\tau)$ satisfy conditions A1, A2, A4, and A5. If $r(t,\tau)$ satisfies A3 on $T \times T$, then

(a) $d_k^2 > 0$, for all t in T,

and (b) $R_k(t) > 0$, for t dense in a neighborhood of t_o.

<u>Proof.</u> The proof of the Theorem follows directly from Corollary 3.3 and Lemma 3.4.

Sufficiency is established in the following Theorem.

<u>Theorem 4.5.</u> Let $T = (t_o, t_1)$ and let $r(t,\tau)$ satisfy conditions A1, A2, A4, and A5. If

(a) $d_k^2(t) > 0$, for all t in T,

and (b) $R_k(t) > 0$, for t dense in a neighborhood of t_o,

then $r(t,\tau)$ satisfies A3 for t and τ in a neighborhood of t_o. If, in addition to (a) and (b) above, some solution $M(t)$ of (3.21) is finite for t in T, then $r(t,\tau)$ satisfies A3 on $T \times T$.

<u>Proof.</u> The hypotheses A1, A2, A4, and A5 insure that the matrix

Riccati equation (3.21) may be formulated for a matrix $M(t)$, and (3.21) is valid on T. If $d_k^2(t) > 0$, then (3.21) implies that $\dot{M}(t) \geq 0$.

Hypothesis (b) and Theorem 3.4 implies that a symmetric initial matrix M_o may be determined at a point t', arbitrarily close to t_o, such that

$$M_o \geq 0 \quad ,$$

and

$$\Gamma_k(t') = M_o \, \Phi_k(t') \quad .$$

Therefore

$$M(t) = M_o + \int_{t'}^{t} \dot{M}(\lambda) \, d\lambda$$

is an admissible matrix. Theorem 3.3 implies

$$\Gamma_k(t) = M(t) \, \Phi_k(t) \quad ,$$

for all t in a neighborhood of t', and in particular,

$$\gamma(t) = M(t) \, \phi(t) \quad . \tag{4.17}$$

Therefore, from A1 and A2,

$$r(t,\tau) = \phi^t(t) \, M[\min(t,\tau)] \, \phi(\tau) \quad ,$$

for t and τ in a neighborhood of t', and from Theorem 2.1, $r(t,\tau)$ satisfies A3 for t and τ in a neighborhood of t'. But t' may be chosen arbitrarily close to t_o. Hence $r(t,\tau)$ satisfies A3 for t and τ in a neighborhood of t_o.

If $M(t)$ is finite on T, then (4.17) is valid everywhere on T. Hence, $r(t,\tau)$ satisfies A3 on $T \times T$.

The results presented above completely characterize the non-negative definite property of the class of functions which satisfy conditions A1 - A5. Note that condition (a) requires only the determination of the sign of a scalar-valued function on an interval. Condition (b) requires that a matrix of order $k+1$ be positive definite in an arbitrarily small neighborhood of a point. An equivalent condition, [4.10], requires that $k+1$ determinants, of orders 1 to $k+1$, be computed, and their signs determined in a small neighborhood of the point t_o.

The present results and Bochner's theorem are similar in the sense that both require the determination of a quantity which in the stationary case is regarded as the power spectrum of the process, and in the non-stationary case, is regarded as the instantaneous power of the white-noise component of the k-th derivative of the process. This similarity, although interesting, is limited as shown in Example 4.5.

The following examples will conclude this section.

Example 4.4. The First-Order Case

Let

$$r(t,\tau) = \begin{cases} \phi(t)\,\gamma(\tau), & \text{for } t \geq \tau \\ \phi(\tau)\,\gamma(t), & \text{for } t < \tau \end{cases},$$

where ϕ and γ are continuously differentiable on $T = (t_o, t_1)$, and ϕ never vanishes indefinitely on T. Assume $r(t_o,t_o) > 0$, and $d_o^2(t) > 0$ on T. Then,

$$d_o^2 = \phi \dot{\gamma} - \dot{\phi} \gamma = \phi^2 \frac{d}{dt}\left(\frac{\gamma}{\phi}\right) .$$

Defining $M = \gamma/\phi$ implies

$$d_o^2 = \phi^2 \dot{M} > 0$$

Therefore,

$$\dot{M}(t) > 0 , \quad \text{for } t \text{ in } T .$$

With M(t) defined above, we have

$$M_o = M(t_o) = \frac{\gamma(t_o)}{\phi(t_o)} = \frac{\gamma^2(t_o)}{r(t_o, t_o)} \geq 0 .$$

Therefore,

$$M(t) = M_o + \int_{t_o}^{t} M(\lambda)\, d\lambda \geq 0 ,$$

and M is non-decreasing on T, so that $r(t,\tau)$ is an autocorrelation function. This analysis confirms the discussion of Section 2.3.

Example 4.5. Let

$$r(t,\tau) = -2\, e^{-|t-\tau|} + 3\, e^{-2|t-\tau|} ,$$

and let

$$\gamma(t) = \begin{bmatrix} -2\, e^{t} \\ 3\, e^{2t} \end{bmatrix} , \text{ and } \phi(t) = \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix}$$

Then $d_o^2(t) = 8$, and $r(t,t) = 1$. Hence, from Theorem 4.5, $r(t,\tau)$ is non-negative definite in some region. $r(t,\tau)$ has the appearance of a stationary autocorrelation function. The Fourier transform of $r(t,\tau)$ may be determined as

$$\frac{16\,\omega^2 - 4}{(\omega^2+1)\,(\omega^2+4)} \quad ,$$

which has both positive and negative values. Therefore from Bochner's Theorem, $r(t,\tau)$ is not non-negative definite, which is an apparent contradiction. Application of the reduction technique in Section 3.3 yields the following Riccati equation for the scalar covariance $M_s(t)$

$$\dot{M}_s(t) = \frac{1}{8}\left(3\,e^{2t} + M_s(t)\,e^{-2t} \right)^2 \quad ,$$

which has the general solution

$$M_s(t) = \frac{\left[\dfrac{M_s(t_o)+13-\sqrt{160}}{M_s(t_o)+13+\sqrt{160}}\, e^{2\sqrt{160}(t-t_o)} - 13 + \sqrt{160}\right] e^{4t}}{1 - \dfrac{M_s(t_o)+13-\sqrt{160}}{M_s(t_o)+13+\sqrt{160}}\, e^{2\sqrt{160}(t-t_o)}} \quad . \tag{4.18}$$

$M_s(t)$ is finite for all $t \geq t_o$ provided that $M_s(t_o) \leq \sqrt{160} - 13 < \infty$. But, from (3.22) and Corollary 3.4, $M_s(t_o) \geq 9$. Hence, any solution of the Riccati equation which is relevant to the factorization problem must have a finite escape time. Therefore $r(t,\tau)$ is non-negative definite in only a finite region which may be determined from (4.18). This conclusion is not inconsistent with Bochner's theorem, since Bochner's theorem applies to $r(t,\tau)$ defined for all t and τ.

4.3 Stability of the Shaping Filter

In this section, a condition for the stability of a shaping filter represented by (2.5) will be developed. Stability is to be defined in the following way.

Definition 4.1. A linear system is L_2 I.B.O. stable if every square-integrable input produces a bounded output.

The following Lemma provides a sufficient condition for L_2 I.B.O. stability.

Lemma 4.1. A linear system is L_2 I.B.O. stable if

$$\int_{t_o}^{t} h^2(t,\tau)\, d\tau \le C \le \infty \; ,$$

for all $t \ge t_o$, where $h(t,\tau)$ is the impulse response of the linear system.

Proof. From the Schwarz inequality,

$$y^2(t) = [\int_{t_o}^{t} h(t,\tau)\, u(\tau)\, d\tau]^2 \le \int_{t_o}^{t} h^2(t,\tau) d\tau \int_{t_o}^{t} u^2(\tau) d\tau \le C \int_{t_o}^{\infty} u^2(\tau) d\tau \, .$$

Therefore, according to the hypothesis of the Lemma, every input u(t) which is square-integrable on the interval $[t_o, \infty)$ will produce an output y(t) which is uniformly bounded on $[t_o, \infty)$.

Although the stability criterion defined above may seem arbitrary and unfamiliar, this criterion was chosen for two reasons. First, the criterion is extrinsic; that is, L_2I.B.O. stability is characterized by the external (input-output) behavior of a system rather than by the internal behavior of a state-variable model of the system. It was mentioned in Section 2.1 that equivalent state-variable models of a linear system preserve the external behavior (impulse response) of the system, but do not preserve such internal properties as stability of the state-variable. In particular, the shaping filter model in (2.5) may have a well-behaved input-output description, but will rarely be well-behaved internally; i.e., the state variable $x(t)$ and the coefficients $\beta(t)$ and $\phi(t)$ will generally be unbounded. Clearly, an extrinsic stability criterion is desirable.

The second reason for choosing the stability criterion in Definition 2.1 is that the L_2I.B.O. stability property may be directly related to appropriate properties of the autocorrelation function $r(t,\tau)$, as the following Theorem indicates.

Theorem 4.6. Let T represent the interval $[t_0, \infty)$, and let $r(t,\tau)$ satisfy conditions A1 - A5 on T and let $M(t)$ be finite for all $t \geq t_0$. Then any shaping filter corresponding to a finite $M(t)$ is L_2I.B.O. stable if $r(t,t) \leq C < \infty$ for all t.

Proof. Since $r(t,\tau)$ satisfies A1 - A5, there is an admissible matrix $M(t)$ satisfying

$$r(t,\tau) = \phi^t(t)\, M[\min(t,\tau)]\, \phi(\tau) \quad ,$$

for all t and τ in T, and by assumption, $M(t)$ is finite. Therefore from

(2.10) and (2.15),

$$\infty > C \geq r(t,t) = \phi^t(t)\, M(t)\, \phi(t) \geq \phi^t(t)\, [M(t) - M(t_o)]\, \phi(t)$$

$$= \int_{t_o}^{t} h^2(t,\tau)\, d\tau \quad ,$$

for all t in T, where the last equality is valid regardless of the choice of initial covariance $M(t_o)$ provided that $M(t_o)$ produces a finite M(t). Hence, from Lemma 4.1, any shaping filter corresponding to $r(t,\tau)$ is L_2 I.B.O. stable.

For time-invariant or periodically varying systems, L_2 I.B.O. stability is equivalent to the more familiar concept of bounded input bounded output (B.I.B.O.) stability. It is well-known [4.11] that a time-invariant (periodic) system is B.I.B.O. stable if and only if all poles (characteristic exponents) of the system have negative real parts.

The impulse response of a time-invariant or periodically-varying system may be written as

$$h(t,\tau) = \sum_{i=1}^{n} a_i(t)\, b_i(t-\tau)\, c_i(\tau)\, e^{p_i(t-\tau)} \quad .$$

For a periodically-varying system, the functions $a_i(t)$ and $c_i(t)$ for i = 1 ... n, are periodic with a common period, the functions $b_i(t-\tau)$ are polynomials in $(t-\tau)$, and the complex constants p_i are the characteristic exponents appearing in conjugate pairs. For a fixed system, the functions $a_i(t)$ and $c_i(t)$ are unity, the functions $b_i(t-\tau)$ are polynomials, and the p_i's are poles.

Then,

$$h^2(t,\tau) = \sum_{i=1}^{2n} \overline{a}_i(t)\, \overline{b}_i(t-\tau)\, \overline{c}_i(\tau)\, e^{\overline{p}_i(t-\tau)} ,$$

where $\overline{a}_i(t)$ and $\overline{b}_i(t)$ are periodic or unity, $\overline{b}_i(t-\tau)$ is a polynomial, and the constants $\overline{p}_i$ are appropriate sums and differences of the constants p_i. Clearly $\mathrm{Re}(\overline{p}_i) < 0$ if and only if $\mathrm{Re}(p_i) < 0$, for all i. The function $h^2(t,\tau)$ may itself be regarded as an impulse response of a time-invariant or periodically varying system which, as is well-known [4.11], is B.I.B.O. stable if and only if

$$\int_{t_0}^{t} h^2(t,\tau)\, d\tau \le C < \infty ,$$

for all t. Therefore, the equivalence of B.I,B.O. and L_2I.B.O. stability for time-invariant and periodically-varying systems is established.

It is possible, using some recent results of Silverman and Anderson [4.12] concerning stability of linear systems, to develop general criteria for B.I.B.O. stability of the shaping filter. However, such criteria have the disadvantage of requiring detailed knowledge of the behavior of the covariance matrix M(t) and the coefficient $\beta(t)$, which thus far, have not been simply related to properties of the given auto-correlation function $r(t,\tau)$. It was shown in Theorem 4.6 that L_2I.B.O. stability may be readily established by inspection of $r(t,t)$. Therefore, the introduction of this unfamiliar definition of stability is appropriate and justified in the present context.

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CHAPTER V

REALIZATION OF THE SHAPING FILTER: SINGULAR CASES

5.0 Introduction

It was convenient in the previous sections to require the index functions $d_k^2(t)$ to be strictly positive on the interval T. In this chapter, the class of autocorrelation functions under consideration is extended by considering cases for which $d_k^2(t) = 0$ at an isolated point t' in T, and for which $d_i^2(t) = 0$ for all t in T, and for $0 \leq i \leq n-1$. In the former case, sufficient conditions are established which guarantee that the Riccati equation has a solution which is continuous at the singular point. In the latter case, a Riccati equation cannot be formulated, but an algebraic solution of the factorization problem is developed.

5.1 Riccati Equation With an Isolated Singular Point

Cases for which $d_k^2 = 0$ at a point are of more than academic interest, since they may frequently be encountered in practice. For example, if the behavior of a periodically fading radio channel is to be simulated, then one might require the coefficient $\phi(t)$ to vanish periodically. It is possible, then, that $d_k^2(t)$ may vanish periodically, and hence, the Riccati equation will have periodically spaced isolated singular points.

The purpose of this section is to examine the solutions of the Riccati equation (3.21) in a neighborhood of a singular point, and to demonstrate that under certain conditions, there exists a solution M(t) of the Riccati equation which may be extended continuously past the singular point. We assume for simplicity of analysis that the Riccati equation has a single isolated singular point at $t = t'$, and that assumptions A1 - A5 hold in the open intervals T_1 and T_2, where $T_1 = (a, t')$ and $T_2 = (t', b)$ and $a < t' < b$. Then $d_k^2(t) > 0$ for all t in T_1 and T_2, and $d_k^2(t') = 0$. We will restrict attention to those solutions M(t) of (3.21) which are finite.

In order to establish the desired results, the intervals T_1 and T_2 must be considered separately. The following Lemmas establish preliminary results which are applicable on T_1.

Lemma 5.1. Let $r(t,\tau)$ satisfy conditions A1 - A5 on T_1 and T_2 and let $d_k^2(t') = 0$. Let $M_o = M(t_o)$, satisfying (3.22), be an initial condition for the Riccati equation for t_0 in T_1, and let M(t) be everywhere finite. Then the solution M(t) is continuous on $[t_0, t')$, and the left-hand limit

$$\lim_{t \to t'} M(t) = M^-$$

exists. Therefore M(t) is uniformly continuous on $[t_0, t']$.

<u>Proof</u>. By assumption, M(t) is finite. Furthermore, M(t) is admissable in $[t_0, t')$, and hence, is monotone. Therefore

$$\lim_{t \to t'} M(t) = M^-$$

exists and is finite. By including the value M^-, M(t) may be defined on the closed interval $[t_0, t']$ and is continuous, and hence, uniformly continuous on that interval.

By slight abuse of the definition of the term solution, M(t) will be called a solution of the Riccati equation for all t in $[t_0, t']$.

In order to show that M(t) may be extended continuously past the singular point t', we will show that the totality of solutions of the Riccati equation are equicontinuous on $[t_0, t']$. Equicontinuity is defined as follows:

<u>Definition 5.1</u>. Let λ assume values in a parameter set Λ, and let $\|\cdot\|$ denote an appropriate matrix norm. A set of (matrix-valued) functions $\{M_\lambda(t)\}$ is <u>equicontinuous</u> at a point τ if, for every $\epsilon > 0$,

there exists a $\delta > 0$ such that

$$\| M_\lambda(\tau) - M_\lambda(t) \| < \epsilon \ ,$$

for t such that $|\tau - t| < \delta$ and for all λ in Λ.

The set $\{M_\lambda(t)\}$ represents the totality of finite solutions of the Riccati equation corresponding to the set of initial conditions $\{M_\lambda(t_o)\}$ which contains covariance matrices satisfying

$$\Gamma_k(t_o) = M_\lambda(t_o)\Phi_k(t_o) \ .$$

It will be ncessary for later development to restrict attention to those matrices $M_\lambda(t_o)$ which are non-singular.

Certainly, the set $\{M_\lambda(t)\}$ is equicontinuous on $[t_o, t')$, since for each λ $M_\lambda(t)$ is continuous on $[t_o, t')$, and from Theorem 4.3, $\dot{M}_\lambda(t)$ is bounded on $[t_o, t' - \epsilon]$ for any $\epsilon > 0$, and the bound on $\dot{M}_\lambda(t)$ is independent of λ.

Since the set $\{M_\lambda(t)\}$ is equicontinuous, the Ascoli lemma [5.1] implies that $\{M_\lambda(t)\}$ has a uniformly convergent subsequence $M_{\lambda i}(t)$, $i = 1, 2, \ldots$. We will digress for a moment and show that if $M(t)$ is the uniform limit of a sequence of solutions $M_{\lambda i}(t)$, then $M(t)$ is also a solution of the Riccati equation (3.21). This result will be useful later in this section. Denote the right-hand side of the Riccati equation by $F[M_\lambda(t), t]$. Then (3.21) becomes

$$\dot{M}_{\lambda i}(t) = F[M_{\lambda i}(t), t]$$

which may be integrated to yield

$$M_{\lambda i}(t) = M_{\lambda i}(t_o) + \int_{t_o}^{t} F[M_{\lambda i}(\tau), \tau]\, d\tau \quad . \tag{5.1}$$

If

$$\lim_{i \to \infty} M_{\lambda i}(t) = M(t)$$

uniformly on $[t_o, t')$, then (5.1) becomes

$$M(t) = M(t_o) + \lim_{i \to \infty} \int_{t_o}^{t} F[M_{\lambda i}(\tau), \tau]\, d\tau \quad . \tag{5.2}$$

Set $M_{\lambda i} = M + A_{\lambda i}$, where $A_{\lambda i}$ vanishes uniformly in t in the limit. Then, expansion of the function $F[M_{\lambda i}(t), t]$ yields

$$\begin{aligned} F[M_{\lambda i}(t), t] &= F[M(t) + A_{\lambda i}(t), t] \\ &= F[M(t), t] + G[A_{\lambda i}(t), t] \quad , \end{aligned}$$

where the function $G[A_{\lambda i}(t), t]$ is bounded and contains only linear and quadratic terms in $A_{\lambda i}$. Therefore the norm of G satisfies

$$\| G[A_{\lambda i}(t), t] \| \le \| A_{\lambda i}(t) \| \, C \quad ,$$

where C is a finite constant, for $\|A_{\lambda i}(t)\|$ sufficiently small. But

$$\begin{aligned} \lim_{i \to \infty} \left\| \int_{t_o}^{t} G[A_{\lambda i}(\tau), \tau] d\tau \right\| &\le \lim_{i \to \infty} \int_{t_o}^{t} \| G[A_{\lambda i}(\tau), \tau] \| \, d\tau \\ &\le \lim_{i \to \infty} C \int_{t_o}^{t} \| A_{\lambda i}(\tau) \| \, d\tau \\ &= 0 \quad , \end{aligned}$$

since $A_{\lambda i}(t)$ vanishes uniformly in t. By applying this result to (5.2), we have

$$\lim_{i \to \infty} \int_{t_o}^{t} F[M_{\lambda i}(\tau), \tau] d\tau = \int_{t_o}^{t} F[M(\tau), \tau] d\tau \quad .$$

Therefore $M(t)$ is a solution of the Riccati equation for t in $[t_o, t']$.

It was shown above that $\{M_\lambda(t)\}$ is equicontinuous for all t in $[t_o, t')$. We now demonstrate that under certain conditions, $\{M_\lambda(t)\}$ is equicontinuous for $t = t'$.

<u>Lemma 5.2</u>. Let $\{M_\lambda(t)\}$ be the totality of finite solutions of the Riccati equation corresponding to the (non-singular) initial conditions $\{M_\lambda(t_o)\}$. If

$$\sup_{t_o \leq t \leq t'} \int_{t_o}^{t} \left| \frac{\gamma^{(k+1)t}(t)\, \phi^{(k+1)}(\tau) - \phi^{(k+1)\, t}(t)\, \gamma^{(k+1)}(\tau)}{d_k^2(\tau)} \right| d\tau = N < \infty \, , \tag{5.3}$$

then the set $\{M_\lambda(t)\}$ is equicontinuous from the left for $t = t'$.

<u>Proof</u>. Let

$$f = \begin{bmatrix} \gamma^{(k+1)} \\ \phi^{(k+1)} \end{bmatrix}$$

and

$$g = \begin{bmatrix} -\phi^{(k+1)} \\ \gamma^{(k+1)} \end{bmatrix} \quad .$$

Let $Z(t)$ be a fundamental matrix solution of the 2n-th order linear differential equation

$$\dot{Z}(t) = \frac{f(t)\, g^t(t)}{d_k^2(t)} Z(t) \quad , \tag{5.4}$$

and partition Z(t) into n-th order square submatrices as

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} .$$

We show first that

$$\lim_{t \to t'} Z(t) = Z^-$$

exists , where the above expression defines Z^-.

Let the 2n-vector z(t) be any column of Z(t), and define a scalar w(t) as

$$w(t) = g^t(t)\, z(t) \quad , \tag{5.5}$$

and a scalar $w_o(t)$ as

$$w_o(t) = g^t(t)\, z(t_o) \tag{5.6}$$

Then (5.4) becomes

$$\dot{z}(t) = \frac{f(t)}{d_k^2(t)}\, w(t) \quad , \tag{5.7}$$

which may be integrated as follows:

$$z(t) = z(t_o) + \int_{t_o}^{t} \frac{f(\tau)}{d_k^2(\tau)}\, w(\tau)\, d\tau \quad . \tag{5.8}$$

Pre-multiplying (5.8) by $g^t(t)$ yields

$$w(t) = w_o(t) + \int_{t_o}^{t} \left[\frac{g^t(t)\, f(\tau)}{d_k^2(\tau)} \right] w(\tau)\, d\tau \quad , \tag{5.9}$$

a Volterra integral equation which must be solved to obtain w(t). According to (5.6), w(t) depends on the choice of initial value $z(t_o)$.

The vector $z(t)$ may be determined from $z(t_o)$ and $w(t)$ by (5.8). By choosing 2n linearly independent initial valuez $z(t_o)$, 2n functions $z(t)$ may be generated which are linearly independent on $[t_o, t]$. Thus, solutions $w(t)$ of (5.9) will determine a fundamental matrix $Z(t)$. We must therefore determine the existence and uniqueness of solutions of (5.9) on the interval $[t_o, t']$.

Denote the kernel of the integral equation (5.9) by

$$K(t,\tau) = \frac{g^t(t)\, f(\tau)}{d_k^2(\tau)} ,$$

and consider the integral operation

$$v(t) = \int_{t_o}^{t} K(t,\tau)\, w(\tau)\, d\tau + w_o(\tau) = Aw .$$

Let $w_1(t)$ and $w_2(t)$ be continuous functions defined on $[t_o, t']$ and let

$$m = \sup_{t_o \le t \le t'} |w_1(t) - w_2(t)|$$

From the hypothesis of the Lemma,

$$\begin{aligned} |Aw_1(t) - Aw_2(t)| &= \left| \int_{t_o}^{t} K(t,\tau)[w_1(\tau) - w_2(\tau)]d\tau \right| \\ &\le \int_{t_o}^{t} |K(t,\tau)|\ |w_1(\tau) - w_2(\tau)|\, d\tau \\ &\le m N < \infty , \end{aligned} \tag{5.10}$$

where, from (5.3),

$$N = \sup_{t_o \le t \le t'} \int_{t_o}^{t} |K(t,\tau)|\, d\tau . \tag{5.10a}$$

However, the existence of a fundamental matrix $Z(t)$ in a neighborhood of the singular point t' is unaffected by multiplication of the coefficient matrix in (5.4) by a scalar constant, since such a transformation of the coefficient matrix merely produces a linear scaling of the time axis. Such a constant may be chosen so that $N < 1$. Thus, from (5.10), the integral operator A is a contraction mapping, and as is well known [5.2], a solution $w(t)$ of the Volterra equation (4.19) exists and is unique on the closed interval $[t_0, t']$. Hence, since $w(t)$ determines columns of $Z(t)$, the left-hand limit

$$\lim_{t \to t'} Z(t) = Z^- \tag{5.11}$$

exists. We now show that the existence of Z^- implies that $\{M_\lambda(t)\}$ is equicontinuous at $t = t'$.

As is well known, [5.3], any solution of the Riccati equation may be expressed in terms of $Z(t)$ as

$$M_\lambda(t) = [Z_1(t) M_\lambda(t_0) + Z_2(t)] [Z_3(t) M_\lambda(t_0) + Z_4(t)]^{-1}, \tag{5.12}$$

where by assumption, the expression in the right-most bracket is nonsingular for all t in $[t_0, t']$. Equation (5.11) implies that, for $1 \leq i \leq 4$,

$$\lim_{t \to t'} Z_i(t) = Z_i^- \tag{5.13}$$

exists, and from Lemma 5.1,

$$\lim_{t \to t'} M_\lambda(t) = M_\lambda^-$$

exists, where the above limits are taken through values $t \leq t'$.

In order to demonstrate that $\{M_\lambda(t)\}$ is equicontinuous at $t=t'$, the difference

$$\| M_\lambda^- - M_\lambda(t) \| \tag{5.14}$$

must be bounded by some quantity which is independent of λ. By performing elementary algebraic operations on (5.12), we obtain the following expression for the difference (5.14):

$$\| M_\lambda^- - M_\lambda(t) \| = \| \{ (Z_1^- - Z_1(t)) M_\lambda(t_o) + (Z_2^- - Z_2(t)) - M_\lambda^- [(Z_3^- - Z_3(t)) M_\lambda(t_o)$$

$$+ (Z_4^- - Z_4(t))] \} \cdot [Z_3(t) M_\lambda(t_o) + Z_4(t)]^{-1} \| . \tag{5.15}$$

Let

$$S_1 = \sup_\lambda M_\lambda^-$$

$$S_2 = \sup_\lambda \sup_{t_o \le t \le t'} [Z_3(t) M_\lambda(t_o) + Z_4(t)]^{-1} ,$$

and

$$S_3 = \sup_\lambda M_\lambda(t_o) .$$

Lemma 5.1 and (5.13) imply that S_1, S_2 and S_3 are finite. For $1 \le i \le 4$, let

$$\| Z_i^- - Z_i(t) \| = \epsilon_i(t) ,$$

where (5.13) implies that

$$\lim_{t \to t'} \epsilon_i(t) = 0 . \tag{5.16}$$

In terms of the quantities just defined, (5.15) may be expressed as

$$\| M_\lambda^- - M_\lambda(t) \| \le \epsilon_1(t) S_2 S_3 + \epsilon_2(t) S_2 + \epsilon_3(t) S_1 S_2 S_3 + \epsilon_4(t) S_1 S_2$$

$$= \epsilon(t) , \tag{5.17}$$

where (5.17) introduces the quantity $\epsilon(t)$. From (5.16),

$$\lim_{t \to t'} \epsilon(t) = 0$$

and moreover, $\epsilon(t)$ is independent of λ. Therefore the set $\{M_\lambda(t)\}$ is equicontinuous at $t = t'$. Since it was established previously that the set is equicontinuous on $[t_0, t')$, it follows that $\{M_\lambda(t)\}$ is equicontinuous for all t in $[t_0, t']$.

The hypothesis (5.3) of Lemma 5.2 is a reasonable requirement to assure equicontinuity of $\{M_\lambda(t)\}$ since

$$K(t,t) = 0 \quad ,$$

for t in $[t_0, t')$ and

$$\lim_{t \to t'} K(t', t)$$

is generally indeterminate. It must be emphasized however, that condition (5.3) is sufficient, but not necessary. For example, the auto-correlation function considered in Example 2.1 fails to satisfy (5.3), although the covariance $M(t)$ in (2.34) is everywhere continuous. The following example illustrates a case for which (5.3) is satisfied.

Example 5.1. Let

$$r(t, \tau) = t^2(1+\tau)\tau^2, \quad \text{for } t > \tau \quad .$$

Then

$$\phi(t) = t^2$$

and

$$\gamma(t) = (1 + t)t^2.$$

The index function $d_o^2(t)$ may be computed as

$$d_o^2(t) = t^4 \quad ,$$

and the kernel $K(t,\tau)$ as

$$K(t,\tau) = \frac{(2t+3t^2)\,2\tau - 2t(2\tau+3\tau^2)}{\tau^4} \; .$$

The Riccati equation corresponding to $r(t,\tau)$ has one singular point located at $t = 0$. If t_o is chosen close to t then $K(t,\tau)$ does not change sign for $t_o \leq \tau \leq t$. Therefore

$$\left| \int_{t_o}^{t} K(t,\tau)\, d\tau \right| = \int_{t_o}^{t} |K(t,\tau)|\, d\tau = 3 \left| 1 + \frac{t(t-2t_o)}{t_o^2} \right| ,$$

which is finite for all $t \leq 0$.

The following Lemma will complete the discussion of the behavior of the Riccati equation on the interval T_1 for which $t \leq t'$.

Lemma 5.3. Let $r(t,\tau)$ satisfy the hypotheses of Lemmas 5.1 and 5.2, and also satisfy (5.3). Assume that

$$\Gamma_k(t)\, R_k^{-1}(t)\, \Gamma_k^{\,t}(t)$$

is continuous at $t = t'$. Let $\{ M_\lambda^- \}$ be the set of matrices satisfying

$$\lim_{t \to t'} M_\lambda(t) = M_\lambda^-$$

where $\{ M_\lambda(t) \}$ is the totality of non-singular solutions of the Riccati equation, and let $\{ M_\xi(t') \}$ be the set of all non-singular matrices satisfying (3.22). Then, the set $\{ M_\lambda^- \}$ is dense in the set $\{ M_\xi(t') \}$.

Proof. Let $M_\xi(t')$ be any covariance matrix satisfying (3.22). From Corollary 3.4, any matrix satisfying (3.22) may be written as $\Gamma_k R_k^{-1} \Gamma_k^t + N$, where N is a covariance satisfying $N\Phi_k = 0$. By assumption, $\Gamma_k R_k^{-1} \Gamma_k^\tau$ and Φ_k are continuous. Hence, given any $\epsilon_5 > 0$, there exists a $\delta > 0$ and a covariance matrix $M_\lambda(t_0)$ satisfying (3.22) such that $\| M_\xi(t') - M_\lambda(t_0) \| < \epsilon_5$, for all t_0 for which $|t_0 - t'| < \delta$. The matrix $M_\lambda(t_0)$ may be regarded as an initial condition associated with a solution $M_\lambda(t)$ of the Riccati equation. From Lemma 5.1,

$$\lim_{t \to t'} M_\lambda(t) = M_\lambda^-$$

and from Lemma 5.2,

$$\| M_\lambda^- - M_\lambda(t_0) \| < \epsilon$$

independent of λ, for $|t_0 - t'| < \delta$.

From the triangle inequality [5.2],

$$\begin{aligned} \| M_\xi(t') - M_\lambda^- \| &\le \| M_\xi(t') - M_\lambda(t_0) \| + \| M_\lambda(t_0) - M_\lambda^- \| \\ &< \epsilon + \epsilon_5 , \end{aligned}$$

where ϵ and ϵ_5 may be made arbitrarily small by letting t_0 approach t'. Hence, the assertion is proved.

We now investigate solutions of the Riccati equation in the interval T_2 where $t \ge t'$, and will show that, for M(t) non-singular, Lemmas 5.1, 5.2, and 5.3 remain valid if the Riccati equation is integrated backwards, i.e., in the direction of decreasing time.

It will be convenient for the present discussion to reformulate the factorization problem in terms of an anti-causal, i.e., purely anticipative, shaping filter. Let the shaping filter be represented by

$$\dot{x}(t) = \beta^{*}(t)\; u(t) \tag{5.18a}$$

$$y(t) = \gamma^{t}(t)\, x(t) \quad , \tag{5.18b}$$

where β^{*}, γ, and x are n-vectors, and u is scalar-valued white-noise. By a method analogous to that employed in Section 2.2, (5.18a) may be integrated backwards to yield the following expression for the covariance of the state-variable x(t).

$$M^{*}[\max(t,\tau)] = E[x(t)\, x^{t}(\tau)] = M^{*}(t_o) + \int_{t_o}^{\max(t,\tau)} \beta^{*}(\lambda)\, \beta^{*t}(\lambda)\, d\lambda \; , \tag{5.19}$$

for t and $\tau \le t_o$. The method used to prove Lemma 2.1 may be applied to the matrix M^{*}, introduced above, to show that M^{*} is non-negative definite and non-increasing.

In terms of M^{*}, the autocorrelation function of y(t) may be expressed as

$$r(t,\tau) = \gamma^{t}(t)\; M^{*}[\max(t,\tau)]\; \gamma(\tau) \quad .$$

But, from condition A1,

$$r(t,\tau) = \begin{cases} \phi^{t}(t)\, \gamma(\tau) & \text{for } t \ge \gamma \\ \phi^{t}(\tau)\, \gamma(t) & \text{for } t < \tau \quad . \end{cases}$$

Therefore, we may identify

$$\phi(t) = M^{*}(t)\; \gamma(t) \tag{5.20}$$

By comparing (5.20) and (2.26), $M(t)$ and $M^*(t)$ are related by

$$M^{-1}(t) = M^*(t) \quad ,$$

provided that $M(t)$ and $M^*(t)$ are non-singular.

A matrix Riccati equation may be formulated for M^* which is similar to (3.21) except that the functions $\phi^{(k+1)}$ and $\gamma^{(k+1)}$ are interchanged. The index functions d_i^2 are unchanged. An initial condition for this Riccati equation must satisfy

$$\Phi_k(t_o) = M^*(t_o)\,\Gamma_k(t_o) \quad ,$$

and if the Riccati equation is integrated backwards, the solution $M^*(t)$ will solve the factorization problem.

The global existence of $M^*(t)$, for $t \le t_o$, follows from considerations analogous to those in Section 4.1.

If the Riccati equation has a singular point at $t = t'$, then Lemma 5.1 implies that the right-hand limit

$$\lim_{t \to t'} M^*(t) = M^{*+}$$

exists and is finite. If $M^*(t)$ is non-singular, then M^{*+} is non-singular, since $M^*(t)$ is non-increasing. We may then define a positive definite matrix M^+ as

$$M^+ = (M^{*+})^{-1} \quad .$$

If the set $\{M_\lambda^*(t)\}$ is the totality of finite solutions of the Riccati equation in the interval T_2 corresponding to non-singular initial conditions, then the analogue of Lemma 5.3 states that the set $\{M_\lambda^+\}$ is dense in the set

$\{M_{\xi}(t')\}$ of non-singular covariance matrices satisfying (3.22). Note that in T_2, the limits of integration in (5.3) must be interchanged, so that $t' \leq t \leq t_o$.

Having established the validity of Lemmas 5.1 - 3 on both intervals T_1 and T_2, the main result of this section is stated below.

Theorem 5.1. Define intervals T_1 and T_2 as $T_1 = (a, t')$ and $T_2 = (t', b)$. Let $r(t,\tau)$ satisfy conditions A1 - A5 on T_1 and T_2, and let $d_k^2(t') = 0$. Furthermore, let $r(t,\tau)$ satisfy (5.3) in T_2.

Let M_o be a non-singular covariance matrix satisfying (3.22). Then there exists an admissible matrix M(t) which is continuous and satisfies

$$\Gamma_k(t) = M(t)\,\Phi_k(t)$$

for all t in $[t_o, b)$, where $M(t_o) = M_o$, for t_o in T_1.

Proof. If M(t) is the solution of the Riccati equation in T_1 corresponding to the initial condition M_o, then M(t) exists and is continuous in $[t_o, t']$ and satisfies $\Gamma_k(t) = M(t)\Phi_k(t)$ for t in $[t_o, t')$. But $\Gamma_k(t)$ and $\Phi_k(t)$ are continuous on $[t_o, t']$, so that

$$\Gamma_k(t') = M^-\,\Phi_k(t') \quad ,$$

where

$$\lim_{t \to t'} M(t) = M^-$$

M^- is non-singular since M(t) is admissible and M_o is non-singular. Lemmas 5.2, 5.3, and the previous discussion imply that the totality of solutions on T_2, $\{M_\lambda(t)\}$ are equicontinuous on T_2, and for any $\epsilon > 0$, there exists a λ such that

$$\| M_\lambda^+ - M^- \| < \epsilon \quad ,$$

where M_λ^+ is the right-hand limit

$$\lim_{t \to t'} M_\lambda(t) = M_\lambda^+ \quad .$$

From the Ascoli lemma, there exists a uniformly convergent subsequence $M_{\lambda i}(t)$ on T_2 such that

$$\lim_{i \to \infty} M_{\lambda i}^+ = M^- \quad ,$$

and $\lim_{i \to \infty} M_{\lambda i}(t)$ converges uniformly to a solution of the Riccati equation on T_2, and

$$\lim_{t \to t'} \lim_{i \to \infty} M_{\lambda i}(t) = M^-$$

Therefore, the solution $M(t)$ has been extended continuously past the singular point $t = t'$. Hence $M(t)$ exists and is continuous for all t in $[t_0, b)$.

It should be emphasized that equicontinuity was established only as a sufficient and convenient means for producing the desired result. Equicontinuity is not necessary. For example, in the first-order case, the unique solution of the factorization problem is given by $M(t) = \gamma^2(t)/r(t,t)$. Since there is only one solution, it is meaningless to try to establish equicontinuity of the totality of solutions by requiring (5.3) to hold. Indeed, for Example 2.1, (5.3) is violated. Clearly, in the first-order case, the necessary and sufficient condition that $M(t)$ be defined and continuous at a singular point is that $\gamma^2(t)/r(t,t)$ be defined and continuous at the singular point.

5.2 Riccati Equation Undefined on an Interval

Assumptions A1 - A5 were shown to be sufficient to insure that the factorization problem has a solution. In particular, assumption A5 allows the formulation of the Riccati equation, a solution of which was shown to provide the desired factorization. However, according to Lemma 3.5, if $d_i^2(t) = 0$ for all t in T, and for $0 \le i \le n-1$, then $d_i^2(t) = 0$ for all t in T, and for all $i \ge 0$. In such a case, it is not possible to define the Riccati equation on T, and the factorization problem must be solved by other means. It will be shown below that the factorization problem may be solved algebraically, and the solution is a covariance matrix M(t), which is constant on the interval T.

In order to motivate the main Theorem of this section, the following Lemmas will be proved.

Lemma 5.4. Let $r(t,\tau)$ satisfy conditions A1 - A4, and let $d_i^2(t) = 0$ for t in T, and for $0 \le i \le k-1$. If the processes $y^{(o)}(t)$, $y^{(1)}(t)$, ..., $y^{(k)}(t)$, are linearly dependent at each t in a subinterval T' of T, then $d_k^2(t) = 0$ for t in T'. Furthermore, if $\phi(t)$ and $\gamma(t)$ are each n times differentiable, then under the previous hypotheses, $d_i^2(t) = 0$ for t in T', and for $0 \le i \le n-1$.

Proof. The proof is a straightforward extension of the proofs of Lemmas 3.4 and 3.5.

As before, let $Y_k(t) = \text{col}\,[y^{(o)}(t), y^{(1)}(t), \ldots y^{(k)}(t)]$.

The assumption of Lemma 5.4 states that there exists a vector a(t) of dimension $k+1$ such that

$$a^t(t)\ Y_k(t) = 0$$

for t in T'. As in the proof of Lemma 3.4, the above expression may be differentiated, and assuming without loss of generality that the coefficient of $y^{(k)}(t)$, $a_k(t) = 1$ for t in T', an expression for $y^{(k+1)}(t)$ may be derived as a linear combination of the processes $y^{(i)}(t)$, for $0 \le i \le k$. Therefore, $y^{(k+1)}(t)$ exists, and from Corollary 3.1, $d_k^2(t) = 0$ on T'.

The above argument implies that there exists vector b(t) of dimension k+2, such that the (k+2)-nd coefficient $b_{k+1}(t)$ is unity on T', and such that

$$b^t(t)\ Y_{k+1}(t) = 0 \quad ,$$

for t in T'. Differentiating the above expression yields an expression for $y^{(k+2)}(t)$. Since $\phi(t)$ and $\gamma(t)$ are continuously differentiable n times by assumption, the function $d_{k+1}^2(t)$ exists, and from Corollary 3.1, must vanish. Continuing the above argument by induction, proves the assertion.

The following Lemma provides a weak converse to Lemma 5.4.

Lemma 5.5. Let $r(t,\tau)$ satisfy conditions A1 - A3, let $\phi(t)$ and $\gamma(t)$, which are not identically zero be continuously differentiable n times, and let $d_i^2(t) = 0$ for t in T, and for $0 \le i \le n-1$. Then, there exists an integer k, where $0 \le k \le n-1$, such that the matrix $R_i(t)$ has rank k+1 for all t in a subinterval T' of T, where $k \leq i \leq n-1$.

Proof. According to Lemma 5.4, if the components of the vector $Y_j(t)$ are linearly dependent at each t in an interval T', then the components of the vector $Y_\ell(t)$, for $\ell \geq j$, are linearly dependent at each t in T'. Let k be the largest integer for which the components of $Y_k(t')$ are linearly independent for some t' in T, such that the components of $Y_{k+1}(t)$ are linearly dependent in a neighborhood of t'. Clearly, such an integer must exist, since by assumption the given random process $y(t)$ does not vanish everywhere on T.

The integer k is thus determined, and the matrix $R_k(t')$ has rank $k+1$ at the point t' in T, and is therefore non-singular. Since $R_k(t)$ is continuous, there exists a subinterval T' of T, continaing the point t', on which $R_k(t)$ is non-singular, and such that $R_{k+1}(t)$ has rank $k+1$ on T'. From the previous Lemma, $R_i(t)$ has rank $k+1$ on T for $k \leq i \leq n-1$.

It will be shown in the following Theorem that the matrix $R_k(t)$ may be used to determine algebraically a solution of the factorization problem.

Theorem 5.2. Let $r(t,\tau)$ satisfy conditions A1 - A4, and assume that the matrices $R_k(t)$ and $R_{k+1}(t)$ have rank $k+1$ everywhere on an interval T'. Let a matrix $M(t)$ be defined as

$$M(t) = \Gamma_k(t) R_k^{-1}(t) \Gamma_k^t(t) \quad .$$

Then $M(t)$ is admissible, and satisfies

$$\Gamma_k(t) = M(t)\,\Phi_k(t)$$

for t in T', and therefore, M(t) provides a solution of the factorization problem.

<u>Proof</u>. Since $R_k(t)$ and $R_{k+1}(t)$ both have rank k+1 on T', then $y^{(k+1)}(t)$ may be expressed as a linear combination of the processes $y^{(i)}(t)$, where $0 \le i \le k$, for t in T'. This linear combination may be expressed as

$$\dot{Y}_k(t) = A(t)\,Y_k(t) \quad , \tag{5.21}$$

for some matrix A(t). Post-multiplying (5.21) by $Y_k^t(\tau)$, and taking the expectation of the result, yields

$$\frac{\partial}{\partial t} R_k(t,\tau) = A(t)\,R_k(t,\tau) \quad , \tag{5.22}$$

where

$$R_k(t,\tau) = E\,[Y_k(t)\,Y_k^t(\tau)]^{\dagger} . \tag{5.23}$$

Transposing (5.22), and noting that

$$R_k^t(t,\tau) = R_k(\tau,t)$$

yields

$$\frac{\partial}{\partial \tau} R_k(t,\tau) = R_k(t,\tau)\,A^t(\tau) \quad . \tag{5.24}$$

For $t > \tau$,

$$R_k(t,\tau) = \Phi_k^t(t)\,\Gamma_k(\tau) \quad ,$$

† The definition of $R_k(t)$ in (3.5) coincides with $R_k(t,\tau)$ defined by (5.23), for $\tau = t$.

and from equation (5.24),

$$\frac{\partial}{\partial \tau} R_k(t,\tau) = \Phi_k{}^t(t)\,\dot{\Gamma}_k(\tau) = \Phi_k{}^t(t)\,\Gamma_k(\tau)\,A^t(\tau) \quad . \qquad (5.25)$$

Since the concluding discussion of Section 2.2 implies without loss of generality, that, the component functions $\phi_1(t), \ldots \phi_n(t)$ may be assumed linearly independent functions on the interval T', (5.25) yields

$$\dot{\Gamma}_k = \Gamma_k A^t \quad . \qquad (5.26)$$

Define M(t) as

$$M(t) = \Gamma_k(t)\, R_k^{-1}(t)\, \Gamma_k(t) \quad . \qquad (5.27)$$

Clearly, M(t) is defined for all t in T'. From the proof of Theorem 3.4, M(t) so defined is a covariance matrix and satisfies

$$\Gamma_k(t) = M(t)\, \Phi_k(t)$$

for all t in T'. In order to show that M(t) is admissible, it must be established that M(t) is non-decreasing. We show below that M(t) is constant for t in T' and is therefore non-decreasing.

Differentiating (5.27) yields

$$\dot{M} = -\Gamma_k R_k^{-1} \dot{R}_k R_k^{-1} \Gamma_k + \dot{\Gamma}_k R_k^{-1} \Gamma_k{}^t + \Gamma_k R_k^{-1} \dot{\Gamma}_k{}^t \quad . \qquad (5.28)$$

However, the derivative $\dot{R}_k(t)$ may be expressed as

$$\dot{R}_k(t) = \frac{d}{dt} R_k(t,t) = \frac{\partial}{\partial t} R_k(t,\tau)\Big|_{\tau=t} + \frac{\partial}{\partial \tau} R_k(t,\tau)\Big|_{\tau=t}$$

$$= A(t)\, R_k(t) + R_k(t)\, A^t(t) \quad . \qquad (5.29)$$

Substituting (5.26) and (5.29) into (5.28) yields

$$\dot{M} = -\Gamma_k R_k^{-1} A R_k R_k^{-1} \Gamma - \Gamma_k R_k^{-1} R_k A^t R_k^{-1} \Gamma_k + \Gamma_k A^t R_k^{-1} \Gamma_k{}^t + \Gamma_k R_k^{-1} A \Gamma_k{}^t$$

$$= 0 \quad ,$$

for all t in T'. Therefore M is an admissible matrix and $r(t,\tau)$ may be expressed as

$$r(t,\tau) = \phi^t(t)\, M \phi(\tau) \tag{5.30}$$

Since M is constant, the shaping filter has the form

$$\begin{aligned} \dot{x} &= 0 \\ y &= \phi^t x \ , \end{aligned} \tag{5.31}$$

where the initial state x_o is a random variable with covariance matrix M defined by equation (5.27), and $x(t) = x_o$. Furthermore, since the rank of M is k+1, the shaping filter may be reduced to one of order k+1. Let S represent the constant unitary matrix which diagonalizes M, so that

$$M = S^t \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} S \ ,$$

where $\Lambda = \text{diag}\,[\lambda_1, \ldots \lambda_{k+1}]$, and the scalars λ_i are the k+1 non-zero eigenvalues of M. Since M is non-negative definite, $\lambda_i > 0$ for $1 \leq i \leq k+1$. Let S_1 represent the submatrix of S which consists of the first k+1 rows of S, and define a vector $\phi^*(t)$ as

$$\phi^*(t) = S_1\, \phi(t) \ .$$

Note that $\phi^*(t)$ consists of k+1 components. One may verify in a straightforward fashion that

$$r(t,\tau) = \phi^t(t)\, M \phi(\tau) = \phi^{*t}(t)\, \Lambda \phi^*(\tau) \tag{5.32}$$

Therefore, the shaping filter may be represented as

$$\dot{x}^* = 0$$
$$y = \phi^{*t} x^* ,$$

and x^* has dimension k+1. The components of the state-vector x^* are random variables x_i^* such that

$$E[x_i^* x_j^*] = \begin{cases} \lambda_i , & \text{for } i=j \\ 0 , & \text{for } i \neq j . \end{cases}$$

If one desires to approximate an arbitrary continuous autocorrelation function $r(t,\tau)$ by one which is separable on a square, $T \times T$, a logical way to proceed might be to utilize the Karhunen-Loeve expansion [5.4]. Define a sequence of random variables ξ_i as

$$\xi_i = (\lambda_i)^{-\frac{1}{2}} \int_T y(t)\, \phi_i(t)\, dt ,$$

where the scalars λ_i are eigenvalues and the functions $\phi_i(t)$ are orthonormal eigenfunctions of the autocorrelation $r(t,\tau)$; i.e.,

$$\int_T r(t,\tau)\, \phi_i(\tau)\, d\tau = \lambda_i\, \phi_i(t) ,$$

The random variables ξ_i are uncorrelated, and

$$E[\xi_i \xi_j] = \delta_{ij} .$$

A random process $y_n(t)$ may be defined as

$$y_n(t) = \sum_{i=1}^{n} \lambda_i^{\frac{1}{2}} \xi_i \phi_i(t),$$

where

$$\underset{n \to \infty}{\text{l.i.m}}\; y_n(t) = y(t) ,$$

uniformly on T. The autocorrelation function of $y_n(t)$ may be expressed as

$$E[y_n(t)\, y_n(\tau)] = r_n(t,\tau) = \sum_{i=1}^{n} \lambda_i \phi_i(t)\, \phi_i(\tau) \tag{5.33}$$

and, by Mercer's Theorem, [5.5],

$$\lim_{n \to \infty} r_n(t,\tau) = r(t,\tau)$$

where the series converges absolutely and uniformly on $T \times T$. The autocorrelation function $r_n(t,\tau)$ is separable on $T \times T$, and will approximate $r(t,\tau)$ with any desired degree of accuracy, provided that n is chosen sufficiently large.

Theorem 5.2 may be applied to the process $y_n(t)$ in order to obtain a factorization of $r_n(t,\tau)$. However, a factorization may be obtained directly by inspection of (5.33). By setting

$$M = \text{diag}\,[\lambda_1, \ldots \lambda_n] ,$$

an autonomous shaping filter may be described as in (5.31). Thus, the Karhunen-Loeve expansion provides a method for approximating an arbitrary autocorrelation function by another which admits factorization. However, the autonomy of the resulting shaping may be a disadvantage, especially for those signal processing applications which require specification of the whitening filter which inverts the operation of the shaping filter.

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CHAPTER VI

CONCLUSIONS

A shaping filter is a linear system which produces at its output a possibly nonstationary random process with a given autocorrelation function if a stationary white noise process is applied at its input. This investigation was concerned with the synthesis of shaping filters corresponding to separable and differentiable autocorrelation functions. The determination of a shaping filter provides a solution of the so-called factorization problem. The investigation may be regarded as an attempt to establish a formal realizability theory relating the existence of a shaping filter to properties which must be satisfied by an autocorrelation function. The realizability theory is based on a set of conditions (A1 - A5) to be satisfied by the autocorrelation function. If the conditions are satisfied, real-valued coefficients of the shaping filter may be determined by solving a matrix Riccati differential equation of order no greater than the order of the shaping filter.

The model chosen to represent the shaping filter is a set of n linear differential equations in "state-variable" form without feedback, and with a single input and single output. Although this form is often unsuitable for practical simulation, its use is justified in establishing the realizability theory because of the relative simplicity it offers to the formulation and solution of the factorization problem. In many cases, known techniques of system theory allow the zero-feedback model to be simulated by more practical configurations.

In order to formulate the Riccati equation leading to the shaping filter, it was necessary to develop and prove new results concerning the mean square differentiability of a random process. A non-negative function $d_k^2(t)$ derived from the given autocorrelation function was shown to govern the differentiability properties of the process. Specifically, $d_k^2(t)$ corresponds to an instantaneous power associated with the white-noise component of the $(k+1)^{st}$ derivative of the given random process. Since the inverse function $d_k^{-2}(t)$ appears as a coefficient of the Riccati equation, the assumption that $d_k^2(t)$ does not vanish guarantees the existence of the Riccati equation. It was demonstrated in Chapter III that there exists a solution of the Riccati equation which, in turn, solves the factorization problem. By utilizing a set of k+1 linear constraints developed in the same chapter, it was shown that the order of the Riccati equation may be reduced from n to n-k-1.

Conditions which are sufficient to insure a global solution of the Riccati equation and hence, of the factorization problem, were developed by placing an upper bound on the initial condition associated with the Riccati equation. Furthermore, it was demonstrated by example that the finite escape time phenomenon and the asymptotic behavior of solutions of the Riccati equation may bear an interesting relation which should be explored in future research.

A particularly important global property of the shaping filter is stability. Stability was defined in the sense that a square-integrable input produces a bounded output. This type of stability is directly related to salient properties of the given autocorrelation function. If the autocorrelation function is uniformly bounded, the shaping filter will be stable in the sense described provided that the conditions for a global solution of the Riccati equation are also satisfied.

One of the most interesting results obtained here provides a characterization of the autocorrelation functions which admit factorization. The characterization employs a set of easily applied criteria which do not depend explicitly on the non-negative definite condition which, as is well-known, must be satisfied by an autocorrelation function.

In Chapter V, the class of autocorrelation functions under consideration was broadened by admitting processes for which $d_k^2(t)$ vanishes at an isolated point, and processes for which $d_k^2(t)$ vanishes for all k and for all t. In the former case, the Riccati equation has an isolated singular point. Sufficient conditions were established allowing the Riccati equation to have a solution which is defined and continuous at the singular point. In the latter case, the factorization problem may be solved by algebraic means. Autocorrelation functions in the latter category include those corresponding to random processes which may be represented exactly by a truncated Karhunen-Loeve expansion.

Thus, a realizability theory has been presented which defines a large class of autocorrelation functions and establishes the existence of the corresponding shaping filters. Moreover, the theory provides a means for determining the coefficients of the shaping filters. The class of autocorrelation functions under consideration is primarily limited by the assumptions of separability and differentiability. Separability is a necessary condition for realizing a shaping filter in state-variable form. Differentiability is only a sufficient condition, although it is a physically reasonable one. Future research may be directed toward weakening the differentiability assumption. For example, progress in this direction might lead to shaping filters with piecewise differentiable, or switched, coefficients.

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